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DIFFERENTIAL EQUATIONS TUTORIAL

FOR LECTURE AND FOR PRACTICAL EXERCISE

Disusun Oleh:

Kulmirzayeva Gulrabo Abduganiyevna
Muhammad Alvan Rizki

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SAMARKAND STATE ARCHITECTURAL AND CONSTRUCTION UNIVERSITY NAMED AFTER MIRZO ULUGBEK

G. A. Kulmirzayeva

DIFFERENTIAL EQUATIONS

TUTORIAL

For lecture and for practical exercise

Reviewed by the board of the Samarkand State University of Architecture and Civil Engineering named after Mirzo Ulugbek (№11) and recommended for publication as a textbook for students (Construction) 60710400 – Ecology and environmental protection (by industry and sector), 60722500 – Geodesy, cartography and cadastre (by function), 60722800 – Cadastre (by type of activity), 60730300 – Civil engineering: construction of buildings and structures, 60730400 – Construction and installation of utilities (by type), 60730500 – Design and operation of water supply and sewerage systems, 60730800 – Road construction (by type of activity) bachelor's courses based on the decision of the board.

Samarkand – 2024

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Kulmirzayeva Gulrabo Abduganievna. Differential equations. (Tutorial). –Samarkand: SamSACU, -2024, -200 pp.

This textbook was written based on the lecture and for practical problems in the discipline “Differential Equations”. Subject “Differential equations”–2024.–198 pp.

The textbook is compiled in accordance with the program of the discipline “Differential Equations” (Section - Ordinary Differential Equations). Basic theoretical information and 115 examples of solving typical problems are given, as well as recommendations for studying the discipline. Examples are provided for independent work, 85 examples and 20 questions for self-test, recommended literature.

This collection was prepared in accordance with the undergraduate curriculum (Construction) in the subject “Differential Equations” in the Republic of Uzbekistan based on existing state educational standards. 60730500 – Design and operation of water supply and sewerage systems, 60730800 – Road construction (by type of activity), 60710400 – Ecology and environmental protection (by industry sector), 60722500 – Geodesy, cartography and cadastre (by function), 60722800 – Cadastre (by type of activity), 60730300 – Civil engineering: construction of buildings and structures, 60730400 – Construction and installation of engineering communications (according to types) provided for by the undergraduate curriculum, Prepared according to the scientific program. Here, the method of successive approximations of solutions to differential equations is considered, the theorem for the existence of a solution to a differential equation and the uniqueness theorem are proved. It includes theoretical information on practical subjects in the field of mathematical sciences, exemplary samples and practical problems for independent solution.

Compiled by: Senior teacher of the department of “Sociology and Natural Science” of SamSACU Kulmirzayeva Gulrabo Abduganievna.

Reviewers: Doctor of Philosophy in Physical and Mathematical Sciences (PhD) of the Department of Sociology and Natural Science Shavkat Zikiryaev.

Head of the Department of Differential Equations at SamSU, Professor Yahyo Mukhtarov.

Reviewed and approved at the meeting of the Department of Sociology and Natural Science (from “______2024. Minutes No.12)

Preface

This textbook is required for the differential equation course for the 2nd course in the following areas: 60710400 – Ecology and environmental protection (by sector), 60722500 – Geodesy, cartography and cadastre (by function), 60722800 – Cadastre (by type of activity), 60730300 – Civil engineering: construction of buildings and structures ,60730400 – Construction and installation of utilities (by type), 60730500 – Design and operation of water supply and sewerage systems, 60730800 – Road construction (by type of activity) of universities because currently in many books on electrical engineering, radio engineering, and automation, the study of solutions to systems of differential equations is carried out using the apparatus of matrix theory.

Here we consider the method of successive approximations for solving differential equations, and prove a theorem on the existence of a solution to a differential equation and a uniqueness theorem.

In this tutorial, students can use the solved examples and can independently solve these examples themselves. With this textbook, students can read lecture topics and can take advantage of practical exercises.

Training engineers who meet modern requirements is impossible without increasing the level of knowledge in mathematics, which is considered fundamental. Therefore, "Higher Mathematics" is of great importance in the formation of a wide range of engineers. In addition, mathematics is a tool for the successful mastery of many technical sciences.

The subject "Differential Equations" is related to technical solutions to problems. Studying modern mathematical methods, helping students acquire the knowledge they acquired after graduating from higher educational institutions, in solving pressing practical issues in their daily activities, as well as studying scientific and methodological literature on modern methods of improving their professional qualifications.

Chapter I.
Differential equations
1-§. Problems leading to differential equations.

Differential equation - this is an equation connecting two or more functionally dependent values of their differentials or, equivalently, derivatives. The problem of composing and solving, as they say, integrating) such equations often arises in physics and technology. [1].

In this chapter we want to consider ordinary differential equations of all orders, so that it is convenient for all students to solve.

When solving many geometric and physical problems, you have to find an unknown function given a relationship between this unknown function, its derivatives and independent variables. Such a relationship is called a differential equation, and finding a function that satisfies the equation is called solving, or integrating, the given equation.

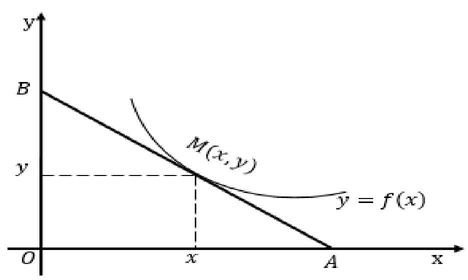
Let the function $y = f(x)$ reflect the quantitative side of some phenomenon. Often, when considering this phenomenon, we cannot directly establish the nature of the dependence of them, but we can establish the relationship between the quantities x and y the derivatives of y x: $y', y'', \ldots, y^{(n)}$, that is, write a differential equation.

From the obtained dependence between the variables x, y and derivatives, it is necessary to establish a direct dependence on x, that is, find $y = f(x)$ or, as they say, integrate the differential equation.

Let's consider several problems leading to differential equations. [10].

Task 1. Find a curve that has the property that a segment of any of its tangents, enclosed between the coordinate axes, is divided in half at the point of tangency.

Let $y = f(x)$ be the equation of the desiredcurve, $M(x, y)$ – arbitrary point at y



In the curve (Fig. 1).

The angular coefficient of the tangent in y $M(x, y)$ $y = f(x)$ this point is equal to y′. According to the condition, x $AM = MB$, that is, a $0PA = x$ means at any point of the M curve $\overline{O} = \overline{P} = x$

(Fig.1) $tg < MAP = -y' = \frac{x}{y}$; there fore, $y' = -\frac{y}{x}$

We have obtained a relationship connecting the unknown function y, the independent variable and the derivative of y, that is, we have obtained a differential equation with respect to. This equation satisfies function $y = \frac{C}{x}$, where C is any number.

Indeed, if $y = \frac{C}{x}$, then $y' = -\frac{C}{x^2}$ and $-\frac{y}{x} = -\frac{C}{x^2}$.

Thus, there are innumerable sets of curves ("family" of curves) differing in the values of the constant C. This is a family of equilateral hyperbolas whose asymptotes are coordinate axes.

In order to select one specific curve from this family of curves, it is enough to specify the point (x_0, y_0) through which this curve passes and determine the corresponding value of the constant C.

For example, through the point (2, 4) there will be a family curve for which $4 = \frac{C}{2}$, that is, $C = 8$. The equation of this curve is $y = \frac{8}{x}$

Problem 2. (radioactive decay problem). It has been established experimentally that the rate of radioactive decay at each moment of time is proportional to the available amount of radioactive substance. It is assumed that the amount of radioactive substance in the rock is so small that it does not cause a chain reaction. It is required to find the law of decay of a substance, that is, to find the dependence of the amount of a radioactive substance on its type on time.

Solution. Let m be the amount of radioactive substance by type at time *t.* The rate of change in the amount of substance is equal. Denoting the positive proportionality coefficient by k, we write the basic law of radioactive decay in the form: $\frac{dm}{dt}$

$$\frac{dm}{dt} = -km$$

(the minus sign is removed because the decay rate is negative $\frac{dm}{dt}$).

The resulting relationship is a differential equation relating the desired derivative function $\frac{dm}{dt}$.

It is easy to verify that any function

$$m = Ce^{-kt}$$

Where C is the number that satisfies this equation.

Really,

$$\frac{dm}{dt} = -kCe^{-kt}, \qquad -km = -kCe^{-kt}$$

that is

$$\frac{dm}{dt} \equiv -km$$

Since C is an arbitrary number, the equation has an infinite number of solutions that differ in the values of the constant C. In order for the problem to become specific, it is enough to indicate the amount of radioactive substance in the rock at some ("initial") moment of time t_0 . Let at $t = t_0$ there were m_0 grams of the substance in the rock. Then the constant corresponding to the solution will be determined from the relationship, and we will obtain the decaylaw in the form: $t_0 m_0 = C_0 e^{-kt_0} C_0 = m_0 e^{kt_0}$

$$m = m_0 e^{-k(t-t_0)}.$$

Using this relationship, you can determine the half-life of substance T, that is, the time during which the amount of the substance will decrease by half. For this we set $m = \frac{m_0}{2}$, $at - t_0 = T$

$$\frac{m_0}{2} = m_0 e^{-kT}, \quad e^{-kT} = \frac{1}{2}, \quad T = \frac{ln2}{k} \approx \frac{0,693}{k}$$

The constant k is assumed to be known.

If the radioactivity coefficient k is unknown, but the half-life of the substance is known, then $k = \frac{ln2}{T}$ and the decay law will be written as:

$$m = m_0 \left(\frac{1}{2}\right)^{\frac{t-t_0}{T}}.$$

Problem 3. A material point of mass m moves in a straight line, attracted to a fixed center by a force proportional to the distance of the point to this center. Find the law of motion of the point.

(Fig.2)

Let's take point 0 as the origin, and the straight line along which the point moves as the OY axis (Fig. 2).

Attractive force $P = -ky$, where y is the coordinate of point m, a k –coefficient of proportionality $(k > 0)$. Since, according to Newton's second law, force is equal to the product of mass m and acceleration, we obtain the differential equation of $\frac{d^2}{dt^2}$ motion of a point:

$$m\frac{d^2y}{dt^2} = -ky$$

This equation relates the desired function y, its second derivative $\frac{d^2}{dt^2}$ and the independent variable - time t. It's easy to check that the function

$$y = C_1 \cos\sqrt{\frac{k}{m}}t + C_2 \sin\sqrt{\frac{k}{m}}t$$

C_1 *and* C_2 for any value of the number satisfies the equation. The constants on which the solution of the equation depends will be determined if specific "initial" conditions of motion are specified.

Let, for example, it be known that at time $t = 0$ the material point was at a distance from point a_0 0 and had a speed v_0.
Then, substituting the relations

$$y = C_1\cos\omega t + C_2 \sin\omega t, \qquad \omega = \sqrt{\frac{k}{m}}$$

$$y' = -C_1\omega\sin\omega t + C_2\omega\cos\omega t$$

instead of t the value $t = 0$, we get $a_0 = C_1$, $v_0 = \omega C_2$,

$$C_1 = a_0, \qquad C_2 = \frac{v_0}{\omega}$$

Thus, the desired law of motion has the form:

$$y = a_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

Believing

$$a_0 = R\sin\alpha, \qquad \frac{V_0}{\omega} = R\cos\alpha,$$

we can write the law of motion as:

$$y = R\sin(\alpha + \omega t)$$

(here $R = \sqrt{a_0^2 + \frac{v_0^2}{\omega^2}}, \; tg\alpha = \frac{a_0\omega}{v_0}$), from which we conclude that the movement in question is a periodic oscillatory movement, R is the amplitude, is the initial phase, is the frequency of oscillation $\omega = \sqrt{\frac{k}{m}}$.

Definition 1. Ordinary differential equation*n* – *th order* called a relationship of the form:

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

where F is a function defined outside a certain region, x is an independent variable, y is the desired function of the variables x, and a are its derivatives $y', y'', \ldots, y^{(n)}$. In this case, the function F can clearly be independent of a, but it must necessarily depend on $y', y'', \ldots, y^{(n-1)} y^{(n)}$.

Definition 2. Order of a differential equation is called the order of the highest derivative included in the equation.
So, for example, equations

$$y' + xy - x^2 = 0, \quad xy'^2 + e^x = 0, \quad yy' - 1 = 0, \quad y'^5 + e^{y^2} = 0$$

will be first-order differential equations, the equations

$$y'' + ky' - by - sinx = 0, \quad xy'' - y'^3 - y = 0, \quad y'' - y' = 1$$

there is a second order equation, equations

$$y^2 - y''' + x^5 = 0$$

has third order, etc.

Definition3. By decisionof a differential equation is any function y = f(x), the substitution of which into this equation turns it into an identity.

For example, the differential equation $y'' + y = 0$ has a solution of the function $y = cosx$, if then $y' = -sinx$, $y'' = -cosx$ and $-cosx + cosx \equiv 0$.

The solution to a differential equation, defined implicitly by the relation $\Phi(x, y) = 0$, is called the integral of this equation.
The graph of the solution to a differential equation is called its integral curve.

Example 1. Let us have the equation $\frac{d^2y}{dx^2} + y = 0$.

Function $y = sinx$, $y = 2cosx$, $y = 3sinx - cosx$ and general functions of the form

$$y = C_1 sinx, \ y = C_2 cosx \ or \ y = C_1 sinx + C_2 cosx$$

are solutions to this equation for any choice of constants C_1 and C_2; This can be easily verified by putting the indicated functions into the equation.

Example 2. Let's consider the equation $y'x - x^2 - y = 0$. Its solutions will be all functions of the form y $= x^2 +$ Cx , where C is any constant. Indeed, differentiating the function $y = x^2 + Cx$, we find $y' = 2x + C$. Substituting the expressions u and y′into the original equation, we obtain the identity

$$(2x + C)x - x^2 - x^2 - Cx = 0$$

Example 3. Solve the equation $x(y^2 - 4)dx + ydy = 0$.

Solution. Dividing the two sides of the equation na $y^2 - 4 \neq 0$, we have

$$xdx + \frac{ydy}{y^2 - 4} = 0$$

Integrating, we find

$$x^2 + ln|y^2 - 4| = ln|C|, \qquad y^2 - 4 = Ce^{-x^2}$$

This is the general solution to this differential equation.

Let now $y^2 - 4 = 0$, that is, $y = \pm 2$.

By direct substitution we verify that $y = \pm 2$ is a solution to the original equation. But it will not be a special solution, since it can be obtained from the general solution with $C = 0$.

Try to decide for yourself [3]

1. Find the partial integral of the equation $y'cosx = \frac{y}{lny}$ satisfying the initial condition $y(0) = 1$.
2. *Find* the general integral of the equation $y' = tgxctgy$.
3. Find a partial solution to a differential equation $(1 + x^2)dy + ydx = 0$ initial condition $y(1) = 1$.
4. Find curves for which the sum of the length of the normal and the subnormal is a constant value equal to a. The length of the subnormal is equal to , and the length of the normal is equal to $|yy'||y\sqrt{1 + y'^2}|$.

Answers.1) $\frac{1}{2}ln^2 y = lntg\left(\frac{x}{2} + \frac{\pi}{4}\right)$

2) $sinycosx = C$ (general integral)

3) $y = e^{\frac{\pi}{4} - arctgx}$

4) +it follows that C takes only positive values.

$$|a^2 - y^2| = a^2 - y^2, that\ is\ y^2 < a^2;$$

2 - §. Basic definitions

The differential equation is obtained as an equation connecting the argument or arguments, the unknown function and its derivatives; Even if initially there was a relationship between differentials, then you can move on to a relationship between derivatives. If the desired function depends on one argument, then the differential equation is called ordinary; otherwise it is called a partial derivative equation. [8].

The highest order of the derivative of the search function included in the equation is called the order of this equation. Thus, equations (1) and (2) are of the first order, while the differential equation for the law of oscillations takes the form

$$M\frac{d^2y}{dt^2} + ky = 0,\ \ y = y(t) = 1 \qquad (1)$$

looks like

$$F(x, y, y', y'', \dots, y^{(n)}) = 0 \qquad (2)$$

where $y = y\ (x)$ is the desired function. Of course, in this case, the function F may not actually depend on all the written values: equation (1) does not include an independent variable and a first-order derivative.

The solution to a differential equation is a function that, when substituted into this equation, reverses its identity.

For example, from the simplest equation

$$y' = x^2,\ \ y = y(x) \qquad (3)$$ we'll

find it right away using integration

$$y = \frac{x^3}{3} + C \qquad (4)$$

This is a general solution to equation (3); it includes an arbitrary constant and is a record of the whole variety of solutions. By giving an arbitrary constant specific numerical values, we obtain specific, particular solutions to equation (3);

$$y = \frac{x^3}{3},\ y = \frac{x^3}{3} + 6, \qquad y = \frac{x^3}{3} - \frac{\sqrt{2}}{3} \quad \textit{and} \quad \textit{o the}$$

In the general case (2), the solution is found as a result of n successive integrations, so that the general solution of an nth order equation contains n arbitrary constants, i.e. looks like

$$y = y(x, C_1, C_2, \dots, C_n) \qquad (5)$$

Especially often the general solution is obtained in an implicit form:

$$\Phi(x, y; C_1, C_2, \dots, C_n) = 0 \qquad (6)$$

Relations (5) and (6) are also called general integrals of equation (2). Particular solutions are obtained by giving each arbitrary constant a specific numerical value. The graph of each particular solution is called the integral line of the differential equation under consideration. The equation of this line is equation (5) and (6) specific $C_1, C_2, \dots, C_n$.

In order to isolate a one-part solution from a general solution, it is necessary, in addition to the differential equation, to set some additional conditions. Most often, initial conditions are set, which, when studying a process developing over time, are a mathematical record of the initial state of the process.

For example, when considering the process of oscillation, that particular oscillation is completely determined if the initial deviation and the initial speed of the oscillating point are given. Therefore, the initial conditions for equation (2) have the form

assigned $t = t_0 \; y = y_0 \quad and \quad \frac{dy}{dt} = v_0$ (7)

In general, for equation (2), the initial conditions have the following form:
at (8) $x = x_0, \; given \; y = y_0, \; y' = (y')_0, \dots, y^{(n-1)} = (y^{(n-1)})_0$

Since the general solution (6) contains n arbitrary, ton of imposed n relations are sufficient, in any case, in principle, to find these constants and thereby to find a particular solution. And it is physically natural that if the differential law governing the development of the process, as well as the initial state of this process, is known, then the process itself is completely determined.

For a first-order equation (3), condition (8) means that for some value the value $y = y_0$ must be specified. Let, for example, it be required that $y(1) = 2$. Then from (4) we obtain $x = x_0$

$$2 = \frac{1^3}{3} + C, \; C = \frac{5}{3},$$

that is, the desired particular solution has the form

$$y = \frac{x^3+5}{3}.$$

The problem of finding a particular solution to a differential equation given an initial condition is called the Cauchy problem.

3-§. First order differential equations

First-order differential equationis called the relationship between the independent variable, the unknown function and its derivative.[9]

1. The first order differential equation has the form

$$F(x, y, y') = 0 \qquad (1)$$

If this equation can be resolved relative to', then it can be written in the form

$$y' = f(x,y) \quad (2)$$

It is called a first order differential equation resolved with respect to the derivative.

A first-order differential equation resolved with respect to the derivative can always be written in the so-called differential form:

$$P(x,y)dx + Q(x,y)dy = 0 \quad (3)$$

Indeed, if $y' = f(x,y)$,

That $\frac{dy}{dx} = f(x,y)$, which means

$$f(x,y)dx - dy = 0$$

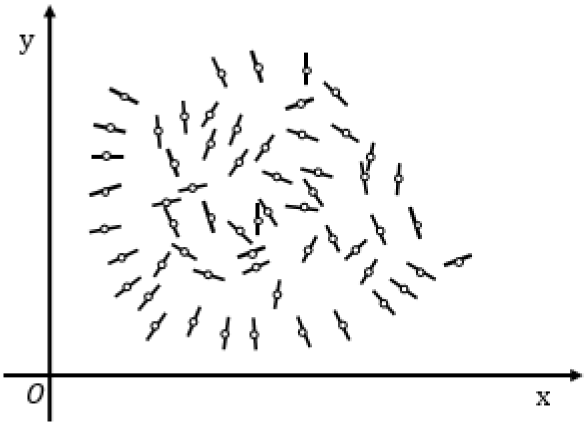
On the contrary, any equation of the form (3), if $Q(x,y) \neq 0$, can be resolved with respect to the derivative:

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}.$$

Let us clarify the geometric meaning of equation (2).

Let xiy be the Cartesian rectangular coordinates of the points of the plane, and $y = \varphi(x)$ be the solution to this equation. The graph of this solution - the integral curve of equation (2) - is a continuous curve, at each point of which there is a tangent. The angular coefficient of the tangent integral curve of the point (x, y) is equal to y′, that is, equal to $f(x,y)$. The equation $y' = f(x,y)$ gives the relationship between the coordinates of the point and the angular coefficient of the tangent integral curve at this point.

At each point (x, y) of the region D in which the function is defined $f(x, y)$, we can calculate y′, that is, indicate the direction of the tangent to the integral curve that passes through this point. By constructing a line (“arrow”) at each point of the region, inclined along the axis OX at an angle tangent equal to $f(x,y)$, we obtain the so-called “field of directions” (Fig. 3).



(Fig.3)

To set the equation $y' = f\ (x, y)$ means to set a direction field in the region D. Finding a solution to this equation means finding a curve whose tangent at each point coincides with the direction of the field at that point.

There will be more than one such curve, but a whole family. To select a specific integral curve, you need to specify a point (x_0, y_0) through which the curve should pass. Under certain restrictions on the right side of equation (2), one integral curve will pass through each point of region D.

Example. Consider the equation $y' = x + y$.

The function $f(x,\ y) = x + y$ is defined everywhere, therefore, the direction field for a given equation can be constructed in the entire plane. In order to organize the arrangement of field directions, we will use the isocline method. An isocline of a direction field is a geometric location of points at which the direction of the field is the same.

Let us denote by α the angle of inclination of the axis OX the direction of the field: $tg\alpha = y'$.

Isocline, at points of which $\alpha = 0,\ i.e.\ y' = tg0 = 0,$
has the equation $x + y = 0.$

Isocline, at points of which, i.e. $y' = tg\frac{\pi}{6} = \frac{\sqrt{3}}{3}$ $\alpha = \frac{\pi}{6}$ has an equation

$$x + y = \frac{\sqrt{3}}{3} \quad etc.$$

In order to draw an integral curve given the direction field, you need to take any point (x_0, y_0) on the XOY plane as the starting point and draw through the line so that at each of its points it goes in the direction of the field.

Note. In practice, the isocline method can be used to approximate the construction of the family of integral curves of equation (2). Moreover, the more isoclines are constructed, i.e., the “more densely” the field directions are indicated on the drawing, the more accurately it is possible to draw the integral curves of the equation.

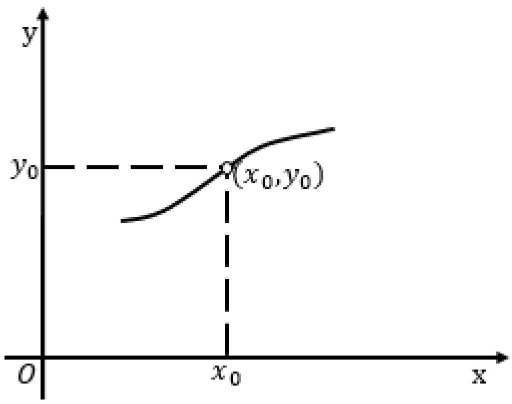
The isocline method allows you to represent the relative position of the integral curves of the equation.

An example can be clarified that the geometric interpretation of a first-order differential equation shows that a first-order differential equation of the form $y' = f\ (x, y)$ has not one, but an infinite number of solutions. In order to select a specific solution from this countless set of solutions, you usually have to set the value of the desired function at y_0 with some argument value x_0.

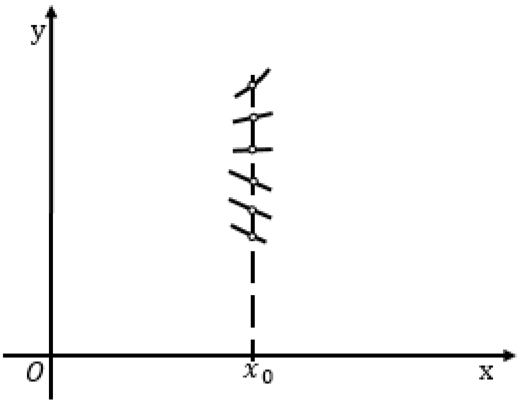
Definition. The pair x_0, y_0 are called initial conditions or initial data of the decision. Geometrically, specifying the initial conditions is equivalent to specifying the point (x_0, y_0) – the "initial point" of the XOY plane. The solution $y_0 = \varphi(x)$ of the equation $y' = f\ (x, y)$ satisfies the initial conditions x_0, y_0, if, that is, if the graph of this $\varphi(x_0) = y_0$ solution passes through the point x_0, y_0.

Finding a solution to the differential equation $y' = f(x, y)$, satisfying the given initial conditions x_0, y_0, is one of the most important problems in the theory of differential equations. This problem is called the Cauchy problem.

Cauchy's theorem. If the function $f(x, y)$ is continuous outside some region D of the XOY plane and has a continuous partial derivative poy in this region y, $f'_y(x, y)$, then whatever the point (x_0, y_0) of the region D is, there exists, and moreover, a unique solution $y = \varphi(x)$ of the equation $y' = f(x, y)$, defined within some interval containing the point x_0, receivingin $x = x_0$ value $\varphi(x_0) = y_0$.



(Fig.4)



(Fig.5)

Geometrically, this statement means that through each internal point (x_0,y_0) of the region D there passes, and only one, integral curve of the equation (Fig. 4).

From Cauchy's theorem it follows that in the domain D the equation $y' = f(x, y)$ has an infinite number of solutions. Indeed, considering 0 constants, changing the value of x_0 beyond certain limits, we obtain for each value of y_0 our solution: $y = \varphi(x, y_0)$ (Fig. 5).

Definition1. The function $y = \varphi(x, C)$, depending on one arbitrary constant C, is called the general solution of the equation $y' = f(x, y)$ outside a certain region if it is a solution to this equation for any value of the constant C and if any solution to the equation lying in the region can be written as $y = \varphi(x, C)$ at a specific value C.

Definition2. The equality $\Phi(x, y, C) = 0$, which implicitly specifies the general solution, is called the general integral of equation (1) in the domain σ.

Definition3. Solutions obtained from the general one at certain values of the constant C are called particular solutions of this equation. Partial integrals are defined similarly.

For such an equation the following theorem is valid, which is called the theorem on the existence and uniqueness of a solution to a differential equation.

Theorem. If in Eq.

$$y' = f(x, y)$$

function f (x, y) and its partial derivative $\frac{\partial f}{\partial y}$ are continuous in some domain D on the Oxy plane containing some point (x_0; y_0), then there is a unique solution to this equation

$$y = \varphi(x)$$

satisfying the condition $y = y_0$ etc $x = x_0$.

Example 1. For a first-order equation $\frac{dy}{dx} = -\frac{y}{x}$, the general solution will be the family of functions $y = \frac{C}{x}$; this can be verified by simply substituting into the equation.

Let us find a particular solution that satisfies the following initial condition: $y_0 = 1$ $x_0 = 2$. Substituting these values of x_0 and y_0 into the formula $y = \frac{C}{x}$, we get $1 = \frac{C}{2}$ or $C = 2$. Consequently, the required partial solution will be the function $y = \frac{2}{x}$.

From a geometric point of view, a general integral is a family of curves on the coordinate plane, depending on one arbitrary constant C. These curves are called integral curves of a given differential equation.

2. Let us give a geometric interpretation of the first order differential equation. Let a differential equation be given that is resolved with respect to the derivative:

$$\frac{dy}{dx} = f(x, y) \qquad (2)$$

and let $y = \varphi(x, C)$ be a general solution to this equation. This general solution defines a family of integral curves in the Oxy plane.

Equation (2) for each point M with coordinates x and y determines the value of the derivative $\frac{dy}{dx}$, that is, the angular coefficient of the tangent to the integral curve passing through this point. Thus, differential equation (2) gives a set of directions or determines the field of directions on the Oxy plane.

Try to decide for yourself [3]

1. $ln cosy dx + x tgy dy = 0$ solve equations.
2. $\frac{yy'}{x} + e^y = 0, \; y(1) = 0$
3. $(1 + e^{2x})y^2 dy = e^x dx; \; y(0) = 0$
4. $y' + cos(x + 2y) = cos(x - 2y); \; y(0) = \frac{\pi}{4}$
5. $y' = 2^{x-y}; \quad y(-3) = -5$

Answers. 1) $y = arccos e^{Cx}$

2) $2e^{-y}(y + 1) = x^2 + 1$

3) $\frac{y^3}{3} + \frac{\pi}{4} = arctg e^x$

4) $ln|tgy| = 4(1 - cosx)$

5) $2^x - 2^y = \frac{3}{32}$

4-§. First-order differential equations integrable by quadratures

Let's consider some of the most important types of first-order differential equations, the integration of which is reduced to finding one or more indefinite integrals. To avoid confusion with the term “integrating an equation,” we will call the action of calculating an indefinite integral a quadrature. [9]

1. Equationsvida $\boldsymbol{y' = f(x)}$, *where* $f(x)$– a function defined continuous over some interval of the $a < x < b$ OX^2 axis.

All solutions to this simplest differential equation are exhausted by the relation

$$y = \int f(x)dx + C \qquad (1)$$

where C is an arbitrary constant.

Geometrically, this means that all integral curves of the equation $y' = f(x)$ in the band {are obtained from one of them, for example, by a shift parallel to the axis of the OY. By specifying any point M_0 (x_0, y_0) in this band, one can uniquely determine the constant C_0 so that the corresponding integral curve

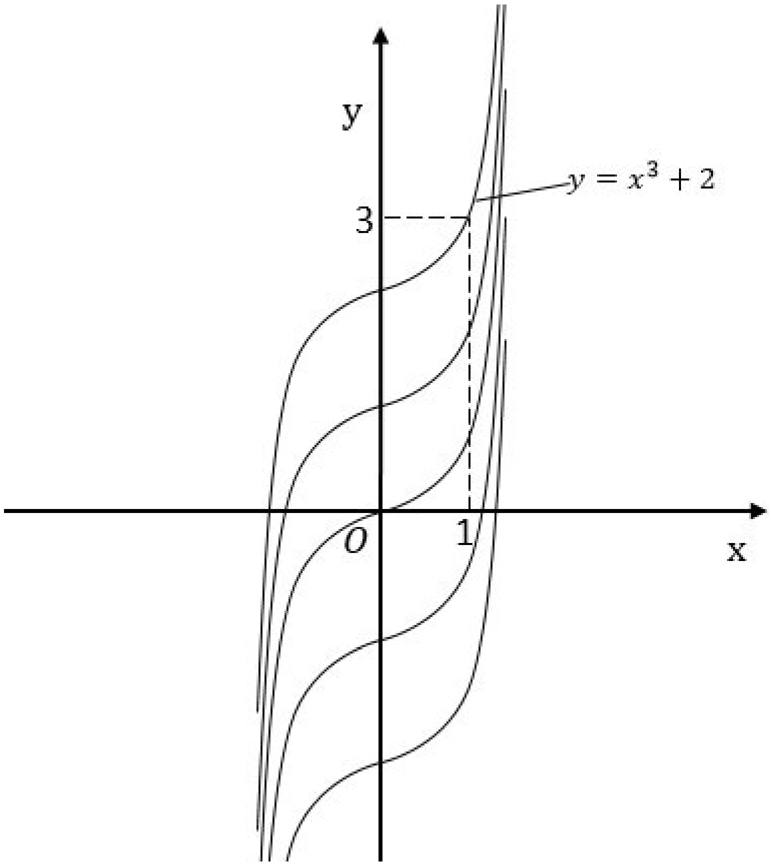
$$\{a < x < b, -\infty < y < +\infty\}\; y = \int f(x)dx$$

$$y = \int f(x)dx + C_0$$

passes through this point.

Relationship (1) is a general solution to the equation $y' = f(x)$ in this strip.

Example. The right side of the equation $y' = 3x^2$ continuous in the interval $-\infty < x < +\infty$.



(Fig.6)

The general solution of the equation in the entire XOY plane has the form $y = x^3 + C$, where C is an arbitrary constant.

Let's find a particular solution that satisfies the initial conditions $x_0 = 1$, $y_0 = 3$. For him $3 = 1 + C_0$, $C_0 = 2$, i.e. $y = x^3 + 2$. Geometrically, this means that from the family of cubic parabolas $y = x^3 + C$, Representing the general solution of the equation, a parabola passing through the point (1, 3) is highlighted - a particular solution of the equation (Fig. 6).

2. Equations with separated variables.

Differential equation type

$$M(x)dx + N(y)dy = 0 \quad (2)$$

called an equation with separated variables. [1]. The general integral of what was proved is

$$\int M(x)dx + \int N(y)dy = C \quad (3)$$

Example 1. Given an equation with separated variables

$$xdx + ydy = 0$$

Integrating, we obtain the general integral:

$$\int xdx + \int ydy = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} = C_1$$

Since the left side of the last equality is non-negative, the right side is also not negative. Denoting $2C_1$ by C_2, we will have

$$x^2 + y^2 = C^2$$

This is the equation of a family of concentric circles with the center at the beginning coordinate and radius C.

3. Equations with separable variables.

Differential equation of the form

$$M_1(x)N_1(y)dx + M_2(x)N_2(y)dy = 0 \quad (4)$$

called an equation with separable variables. It can be compared to a separated variable equation by dividing both sides by the expression $N_1(y)M_2(x)$:

$$\frac{M_1(x)N_1(y)}{N_1(y)M_2(x)}dx + \frac{M_2(x)N_2(y)}{N_1(y)M_2(x)}dy = 0$$

or

$$\frac{M_1(x)}{M_2(x)}dx + \frac{N_2(y)}{N_1(y)}dy = 0 \qquad (5)$$

that is, an equation of the form (2).

Example 2. The equation $y' = x(y^2 + 1)$ is an equation with separable variables. The function $\varphi(x) = x$ and $\psi(y) = y^2 + 1$ is continuous everywhere,

$$y^2 + 1 \neq 0$$

Solution. Separating Variables

$$\frac{dy}{y^2 + 1} = xdx$$

integrating, we get:

$$arctgy = \frac{x^2}{2} + C \qquad (6)$$

the general integral of this equation throughout the XOY plane.

Resolving relation (6) with respect to y, we find a general solution to the equation in the form

$$y = tg\left(\frac{x^2}{2} + C\right), \qquad -\frac{\pi}{2} < \frac{x^2}{2} + C < \frac{\pi}{2}$$

By specifying any initial conditions x_0, y_0, it is possible to determine C_0 from relation (6):

$$arctg\, y_0 = \frac{x_0^2}{2} + C_0, \quad C_0 = arctg\, y_0 - \frac{x_0^2}{2}$$

and, therefore, determine the corresponding partial integral of this equation:

$$arctg\, y = \frac{x^2}{2} + arctg\, y_0 - \frac{x_0^2}{2}$$

and private solution:

$$y = tg\left(\frac{x^2}{2} + arctgy_0 - \frac{x_0^2}{2}\right). \qquad \blacksquare$$

Example 3. Solve the equation $x(y^2 - 4)dx + ydy = 0.$

Solution. Dividing the two sides of the equation na $y^2 - 4 \neq 0$, we have

$$xdx + \frac{ydy}{y^2 - 4} = 0.$$

Integrating, we find

$$x^2 + ln|y^2 - 4| = ln|C| \quad \text{or} \quad y^2 - 4 = Ce^{-x^2}$$

This is the general solution to this differential equation.

Let's compare the equation to zero. From this, $y^2 - 4 = 0,\ y = \pm 2$

By direct substitution we verify that $y = \pm 2$ is a solution to the original equation. But it will not be a special solution, since it can be obtained from the general solution at C = 0. ■

Example 4. The equation is given $\frac{d}{dx} = -\frac{y}{x}$

Solution. We separate the variables: $\frac{dy}{y} = -\frac{dx}{x}$

Integrating, we find

$$\int \frac{dy}{y} = -\int \frac{dx}{x} + C,$$

that is,

$$\ln|y| = -ln|x| + ln|C| \quad and \quad |y| = ln\left|\frac{C}{x}\right|$$

from here we get the general solution: $y = \frac{C}{x}$ ■

Example 5. Given the equation $(1 + x)ydx + (1 - y)xdx = 0.$

Solution. Separating the variables, we find

$$\frac{1+x}{x}dx + \frac{1-y}{y}dy = 0,$$

$$\left(\frac{1}{x} + 1\right)dx + \left(\frac{1}{y} - 1\right)dy = 0$$

Integrating, we get

$$ln|x| + x + ln|y| - y = C \;\; and \;\; |x| + x - y = C;$$

The last relation is the general integral of this equation.

Example 6. Find a partial solution to the differential equation

$(1 + x^2)dy + ydx = 0$ initial condition $y(1) = 1.$

Solution. Let's transform this equation to the form

$$\frac{dy}{y} = -\frac{dx}{1 + x^2}$$

Integrating, we get

$$\int \frac{dy}{y} = -\int \frac{dx}{1+x^2} \quad \text{or} \quad ln|y| = -arctgx + C$$

This is the general integral of this equation.

We use the initial conditions and find an arbitrary constant C; we have

$$ln1 = -arctg1 + C$$

$$C = -\frac{\pi}{4}$$

$$ln\, y = -arctg\, x + \frac{\pi}{4}$$

we obtain the desired particular solution

$$y = e^{\frac{\pi}{4} - arctgx}. \quad \blacksquare$$

Try to decide for yourself [3].

1. $y' + sin(x + y) = sin(xy)$

2. $yy' = -2x\, sec\, y$

3. $y' = e^{x+y} + e^{x-y};\ y(0) = 0$

4. $y' = sh(x + y) + sh(x - y)$

5. $y' = \sqrt{\frac{a^2-y^2}{a^2-x^2}}$

Answers.

1) $2sinx + ln\,|tg\,\frac{y}{2}| = C$

2) $x^2 + ysiny + cosy = C$

3) $y = lntg\,(e^x + \frac{\pi}{4} - 1)$

4) $y = lntg(chx + C)$

5) $y = asin\,(arcsin\frac{x}{a} + C)$;

the answer can also be written in the form $y\sqrt{a^2 - x^2} - x\sqrt{a^2 - y^2} = C_1$

5-§. Homogeneous equations of the first order

Definition 1.The function $f(x,\ y)$ is called a homogeneous function of the nth dimension with respect to the xy variables, if for any $\boldsymbol{\lambda}$ the identity is true [9]

$$f(\lambda x, \lambda y) = \lambda^n f(x,y) \qquad (1)$$

For example, $f(x,y) = x^3 + 3x^2y$ is a homogeneous function of the third dimension relative to xy, since

$$f(tx,ty) = (tx)^3 + 3(tx)^2(ty) = t^3(x^3 + 3x^2y) = t^3f(x,y)$$

Functions

$$f(x,y) = \frac{x^3+y^3}{x^2+xy+y^2}, \quad \varphi(x,y) = \frac{xy}{x+2y}, \quad \psi(x,y) = \frac{x^3}{y} + y^3 ln\frac{x}{y}$$

are homogeneous functions of the first, zero and second dimensions, respectively.

The functions, , are not homogeneous, since for them conditions (1) are not satisfied at any time

$$x^3 - 3x^2y + y, \quad e^{x-y} + 2, \qquad x sin\frac{x}{y} + x^2$$

Homogeneous functions have the following properties:

1. The sum of homogeneous functions of the same dimension is a homogeneous function of the same dimension.
2. The product of a homogeneous function is a homogeneous function whose dimension is equal to the sum of the dimensions of the factors.
3. The quotient of homogeneous functions is a homogeneous function. Its measurement is equal to the difference between the dimensions of the dividend and the divisor.

Example 1. Function $f(x,y) = \sqrt[3]{x^3 + y^3}$ is a homogeneous function of the first dimension, since [1]

$$f(tx,ty) = \sqrt[3]{(tx)^3 + (ty)^3} = t\sqrt[3]{x^3 + y^3} = tf(x,y).$$

Example 2. $f(xy) = xy - y^2$ is a homogeneous function of the second dimension, since

$$(tx)(ty) - (ty)^2 = t^2 (xy - y^2)$$

Example 3. $f(x,y) = \frac{x^2-y^2}{xy}$ is a homogeneous function of zero dimension, since

$$\frac{(tx)^2-(ty)^2}{(tx)(ty)} = \frac{x^2-y^2}{xy}$$

that is

$$f(tx,ty) = f(x,y) \; or \; f(tx,ty) = t^0 f(x,y).$$

Definition 2. First order equation [1]

$$\frac{dy}{dx} = f(x,y) \quad (2)$$

is called homogeneous relative x and y if the function $f\,(x,y)$is a homogeneous function of zero dimension relative x and y.

Any equation of the form will also be homogeneous: [9]

$$P(x,y)dx + Q(x,y)dy = 0,$$

where $P(x,y)$ *and* $Q(x,y)$ are homogeneous functions of the same dimension. Resolving the equation with respect to y′ (or′).

Integrating a homogeneous equation

$$y' = f(x,y) \quad (2')$$

Using a special substitution, it is reduced to the integration of an equation with separable variables.

Indeed, $f(x,y)$ is a homogeneous function of the zero dimension, then for any t

$$f(tx,ty) = f(x,y).$$

Putting in this identity $t = \frac{1}{x}$,we get

$$f\,(x,y) = f\,(1,\frac{y}{x}),$$

This means that the right side of equation (4) actually depends on one argument $\frac{y}{x}$ - the relationship:

$$f(x,y) = \varphi\,(\frac{y}{x}) \quad (3)$$

i.e. equation (3) can be written as:

$$\frac{dy}{dx} = f(1, \frac{y}{x}) \qquad (4)$$

Enter a new unknown function u using the substitution $y = ux$,
Instead of (3), we obtain the equation $u = \frac{y}{x}$

$$u'x + u = \varphi(u)$$

or, also, the equation

$$u' = \frac{\varphi(u) - u}{x} \qquad (5)$$

This is an equation with separable variables for an unknown function u.

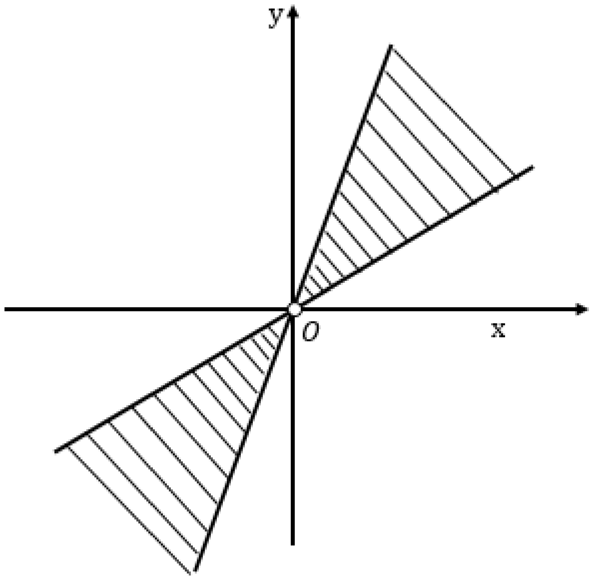
Suppose that the function $\varphi(u)$ is continuous on the interval $a < u < b$ and $\varphi(u) - u \neq 0$.

Separating the variables in equation (5) and integrating, we find the general integrals of this equation in the areas $\{a < u < b, x > 0\}$ and $\{a < u < b, x < 0\}$ in the form:

$$\int \frac{du}{\varphi(u) - u} = \int \frac{dx}{x} + C$$

where C is an arbitrary constant.

By replacing the auxiliary function u with its expression through x and y, we find in quadratures the general integrals of this equation in the areas $\{a < \frac{y}{x} < b,\ x > 0\}\ and\ \{a < \frac{y}{x} < b,\ x < 0\}$ (fig. 7)



(Fig. 7)

Solution of a homogeneous equation.

Condition $f(\lambda x, \lambda y) = f(x, y)$. Putting in this identity we get $\lambda = \frac{1}{x}$

$$f(x, y) = f\left(1, \frac{y}{x}\right)$$

that is, a homogeneous zero-dimension function depends only on the relationship of the arguments.

The equation

$$\frac{dy}{dx} = f(x, y)$$

in this case it will take the form

$$\frac{dy}{dx} = f\left(1, \frac{y}{x}\right)$$

Let's make the substitution $u = \frac{y}{x}$ that is, $y = ux$.

Then we will have

$$\frac{dy}{dx} = u \frac{du}{dx} x .$$

Substituting this derivative expression into equation (2), we get

$$u + x \frac{du}{dx} = f(1, u)$$

This is a separable equation:

$$x\frac{du}{dx} = f(1,u) \ \ and \ \ \frac{du}{f(1,u)-u} = \frac{dx}{x}$$

integrating, we find

$$\int\frac{du}{f(1,u)-u} = \int \frac{dx}{x} + C$$

Substituting the relation instead of u after integration $\frac{y}{x}$, we obtain the integral of equation (4)

Example 4. Given equation $\frac{dy}{dx} = \frac{xy}{x^2-y^2}$. On the right is the homogeneous function of the zero dimension; therefore, we have a homogeneous equation. Making a replacement

$$\frac{y}{x} = u; \text{ then } y = ux, \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + \frac{du}{dx} = \frac{u}{1-u^2}, \qquad x\frac{du}{dx} = \frac{u^3}{1-u^2}$$

Separating the variables, we will have

$$\frac{(1-u^2)du}{u^3} = \frac{dx}{x} \quad \left(\frac{1}{u^3} - \frac{1}{u}\right)du = \frac{dx}{x}$$

from here, integrating, we find

$$-\frac{1}{2u^2} - ln|u| = ln|x| + \ln|C| \ \ or \ \ \frac{1}{2u^2} = ln|uxC|$$

Substituting $u = \frac{y}{x}$, we obtain the general integral of the original equation:

$$-\frac{x^2}{2y^2} = ln|Cy|$$

In this case, it is impossible to obtain an explicit function otx written using elementary functions. However, it is easy to express it here:

$$x = y\sqrt{-2ln|Cy|}. \qquad ■$$

An equation of the form

$$(x,y)\,dx + N(x,y)dy = 0$$

will be homogeneous only if $M(x, y)$ *and* $N(x, y)$ are homogeneous functions of the same dimension. This follows from the fact that the ratio of two homogeneous functions of the same dimension is a homogeneous function of the zero dimension.

Example 5. Equations

$$(2x + 3y)dx + (x - 2y)dy = 0, \quad (x^2 + y^2)dx - 2xydy = 0$$

are homogeneous.

Example 6. The equation is homogeneous. Function $y' = \frac{y}{x}(ln\frac{y}{x} + 1)$

$$f(x,y) = \frac{y}{x}(ln\frac{y}{x} + 1)$$

defined in the region $\{x < 0, y < 0\}$ *and* $\{x > 0, y > 0\}$ (*there* $\frac{y}{x} > 0$, *i.e.* $ln\frac{y}{x}$ makes sense).

We believe $\frac{y}{x} = u$, $y = ux$. Moreover

$$y' = u'x + u, \ u'x + u = u(lnu + 1)$$

$$u'x = ulnu$$

- equation with separable variables with respect to u. Solving it in the regions $\{u > 0, x > 0\}$ *and* $\{u > 0, x < 0\}$, we get:

$$\frac{du}{ulnu} = \frac{dx}{x},$$

$$ln|lnu| = ln|x| + ln|C|, \ C \neq 0,$$

$$lnu = Cx, \ u = e^{Cx}$$

Substituting $u = \frac{y}{x}$ we find $\frac{y}{x} = e^{Cx}$, $y = xe^{Cx}$ the set of solutions to this equation. Here C is any non-zero number.When separating the variables $u = 1$, i.e., the solution is lost:

$$y = x.$$

Since it can be obtained in the $y = xe^{Cx}$ form of $C = 0$, we conclude that, where C is any number, is the general solution of this equation in the regions $\{x > 0, y > 0\}$ *and* $\{x < 0, y < 0\}$.■

Try to decide for yourself [3]

1. Find the general integral of the equation $(x^2 + 2xy)dx + xydy = 0$.
2. Find a particular solution to the equation $y' = \frac{y}{x} + \sin\frac{y}{x}$ under the initial condition $y(1) = \frac{\pi}{2}$
3. $xy' \sin\frac{y}{x} + x = y\sin\frac{y}{x}$
4. $xy + y^2 = (2x^2 + xy)y'$
5. $xy' \ln\frac{y}{x} = x + y\ln\frac{y}{x}$
6. $xyy' = y^2 + 2x^2$
7. $xy' - y = xtg\frac{y}{x};\ y(1) = \frac{\pi}{2}$
8. $(x^2 + y^2)dx - xydy = 0$
9. $y' = \frac{x+y}{x-y}$
10. $xy' = 2(y - \sqrt{xy})$

Answers. 1) $ln|x + y| + \frac{x}{x+y} = C$

2) $y = 2xarctgx$

3) $Cx = e^{\cos\frac{y}{x}}$

4) $y^2 = Cxe^{-\frac{y}{x}}$

5) $lnx = \frac{y}{x}[ln\frac{y}{x} - 1] + C$

6) $y^2 = 4x^2lnCx$

7) $y = xarcsinx$

8) $y^2 = x^2lnCx^2$

9) $arctg\frac{y}{x} = lnC\sqrt{x^2 + y^2}$

10) $16xy = (y + 4x - Cx^2)^2$

6 - §. Equations reduced to homogeneous

Similar equations are represented by equations of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1} \qquad (1)$$

If $c_1 = c = 0$, then equation (1) is obviously homogeneous. Let now c_1 (or one of them) be different from zero. [1]. Let's make a replacement of variables

$$x = x_1 + h, \quad y = y_1 + k \qquad (2)$$

Then

$$\frac{dy}{dx} = \frac{dy_1}{dx_1}$$

Substituting in equation (1) expressions $x, y,$ we will have $\frac{dy}{dx}$

$$\frac{dy_1}{dx_1} = \frac{ax_1+by_1+ah+bk+c}{a_1x_1+b_1y_1+a_1h+b_1k+c_1} \quad (3)$$

Let us choose hk so that the equalities are satisfied

$$\begin{cases} ah + bk + c = 0 \\ a_1h + b_1k + c_1 = 0 \end{cases} \quad (4)$$

that is, we define hik as solutions to the system of equations (4). Under this condition, equation (3) becomes homogeneous:

$$\frac{dy_1}{dx_1} = \frac{ax_1 + by_1}{a_1x_1 + b_1y_1}$$

Having solved this equation and passing again using formulas (2), we obtain a solution to equation (1).

System (4) has no solution if

$$\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$$

that is, $ab_1 = a_1b$. But in this case
$\frac{a_1}{a} = \frac{b_1}{b} = \boldsymbol{\lambda}$ that is, and therefore, equation (1) can be transformed into the form
$a_1 = \lambda a\,,\ b_1 = \lambda b$

$$\frac{dy}{dx} = \frac{(ax+by)+c}{\lambda(ax+by)+c_1} \quad (5)$$

Then by substitution

$$z = ax + by \quad (6)$$

the equation is reduced to an equation with separable variables.

Really,

$$\frac{dz}{dx} = a + b\frac{dy}{dx}$$

where

$$\frac{dy}{dx} = \frac{1}{b}\frac{dy}{dx} - \frac{a}{b} \qquad (7)$$

Substituting expressions (6) and (7) into equation (5), we obtain

$$\frac{1}{b}\frac{dz}{dx} - \frac{a}{b} = \frac{z + c}{\lambda z + c_1}$$

and this is an equation with separable variables.

The technique applied to the integration of equation (1) is applied to the integration of the equation

$$\frac{dy}{dx} = f\left(\frac{ax + by + c}{a_1x + b_1y + c_1}\right)$$

where f is any continuous function.

Example 1. Given equation

$$\frac{dy}{dx} = \frac{x + y - 3x}{-y - 1}$$

To transform its hydrogen equation, we make the substitution $x = x_1 + h,\ y = y_1 + k$. Then

$$\frac{dy_1}{dx_1} = \frac{x_1 + y_1 + h + k - 3}{x_1 - y_1 + h - k - 1}$$

Solving a system of two equations

$$\begin{cases} h + k - 3 = 0 \\ h - k - 1 = 0 \end{cases}$$

we find

$$h = 2, \quad k = 1$$

As a result, we obtain a homogeneous equation

$$\frac{dy_1}{dx_1} = \frac{x_1 + y_1}{x_1 - y_1}$$

which we solve by substitution

$$\frac{y_1}{x_1} = u;$$

Then

$$y_1 = ux_1, \quad \frac{dy_1}{dx_1} = u + x_1\frac{du}{dx_1},$$

$$u + x_1\frac{du}{dx_1} = \frac{1+u}{1-u}$$

and we get an equation with separable variables

$$x_1\frac{du}{dx_1} = \frac{1+u^2}{1-u}$$

Separate variables:

$$\frac{1-u}{1+u^2}\,du = \frac{dx_1}{x_1}$$

Integrating, we find

$$arctgu - \frac{1}{2}ln(1+u^2) = ln|x_1| + ln|C|$$

$$arctg\,u = ln\,|Cx_1\sqrt{1+u^2}|$$

or

$$Cx_1\sqrt{1+u^2} = e^{arctgu}$$

Substituting here instead of u, we get $\frac{y_1}{x_1}$

$$C\sqrt{x_1^2+y_1^2} = e^{arctg\frac{y_1}{x_1}}$$

Finally, passing to the *x*and *y* variables, we finally get

$$C\sqrt{(x-2)^2+(y-1)^2} = e^{arctg\frac{y-1}{x-2}}$$

Example 2. The equation

$$y' = \frac{2x+y-1}{4x+2y+5}$$

it is no longer possible to solve by substituting $x = x_1 + h,\ y = y_1 + k,$ since in this case the system of equations used to determine h and k is unsolvable (here the determinant of the coefficients $\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$ of the variables is equal to zero).

This equation can be reduced to an equation with separable variables by replacing

$$2x + y = z.$$

Then $y' = z' - 2,$ and the equation is reduced to the form

$$z' - 2 = \frac{z-1}{2z+5}$$

or

$$z' = \frac{5z+9}{2z+5}$$

Solving it, we will find

$$\frac{2}{5}z + \frac{7}{25}ln|5z+9| = x + C$$

Since $z = 2x + y,$ we will finally obtain a solution to the original equation in the form

$$\frac{2}{5}(2x+y) + \frac{7}{25}ln|5(2x+y)+9| = x + C$$

$$\frac{2}{5}(2x+y) + \frac{7}{25}ln|10x+5y+9| = x + C$$

or

$$10y - 5x + 7\,ln|10x+5y+9| = C_1$$

that is, in the form of an implicit function uotx.

Try to decide for yourself [3]

1. Find the general integral of the equation
$$(2x + y + 1)dx + (x + 2y - 1)dy = 0$$

2. Find the general integral of the equation
$$(x + y + 2)dx + (2x + 2y - 1)dy = 0$$

3. $2(x + y)\,dy + (3x + 3y - 1)dx = 0$

4. $(x - 2y + 3)dy + (2x + y - 1)dx = 0$

5. $(x - y + 4)dy + (x + y - 2)\,dx = 0$

Answers.

1) $x^2 + y^2 + xy + x - y = C_1$, (it's supposed to be here $C_1 = C^2 - 1$)

2) $x + 2y + 5ln|x + y - 3| = C$

3) $3x + 2y - 4 + 2ln|x + y - 1| = 0$

4) $x^2 + xy - y^2 - x + 3y = C$

5) $x^2 + 2xy - y^2 - 4x + 8y = C$

7 - §. Bernoulli equation

Consider an equation of the form [1]

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad (1)$$

where P(x) and Q(x) are continuous functions of otx (or constants), and $n \neq 0$ and $n \neq 1$ (otherwise the result was a linear equation). This equation, called the Bernoulli equation, is reduced to linear by the following transformation.

Dividing all terms of the equation by, we gety^n

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q \qquad (2)$$

Let us next make the replacement $z = y^{-n+1}$

Then

$$\frac{dz}{dx} = (-n+1)y^{-n}\frac{dy}{dx}$$

Substituting these values into equation (2), we will have the linear equation

$$\frac{dz}{dx} + (-n+1)Pz = (-n+1)Q$$

Having found its general integral and substituting the expression for zy^{-n+1}, we obtain the general integral of the Bernoulli equation.

Example 1. Solve equation $\frac{dy}{dx} + xy = x^3y^3$ (3)

Solution. Dividing all terms by, we get y^3

$$y^{-3}y' + xy^{-2} = x^3 \quad (4)$$

Let's introduce a new function $z = y^{-2}$; then $\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$

Substituting these values into equation (4), we obtain the linear equation

$$\frac{dz}{dx} - 2xz = -2x^3$$

Let us find the general integral:

$$z = uv, \quad \frac{dz}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

We substitute expressions z and $\frac{d}{dx}$ into equation (5):

$$u\frac{dv}{dx} + v\frac{du}{dx} - 2xuv = -2x^3$$

or

$$u\left(\frac{dv}{dx} - 2xv\right) + v\frac{du}{dx} = -2x^3$$

We equate the expression in parentheses to zero:

$$\frac{dv}{dx} - 2xv = 0, \quad \frac{d\frac{v}{}}{v} = 2xdx$$

$$ln|v| = x^2, \quad v = e^{x^2}$$

to determine u we obtain the equation

$$e^{x^2}\frac{du}{dx} = -2x^3$$

Separate variables:

$$du = -2e^{-x^2}x^3dx, \quad u = -2\int e^{-x^2}x^3dx + C$$

integrating by parts, we find

$$u = x^2e^{-x^2} + e^{-x^2} + C, \qquad z = uv = x^2 + 1 + Ce^{x^2}$$

Therefore, the general integral of this equation is

$$y^{-2} = x^2 + 1 + Ce^{x^2}$$

or

$$y = \frac{1}{\sqrt{x^2+1+Ce^{x^2}}}. \quad \blacksquare$$

Example 2. Integrate equation $y'cos^2x + y = tgx$ initial condition $y(0) = 0$.

Solution. We integrate the corresponding homogeneous equation

$$y'cos^2x + y = 0;$$

dividing the variables, we get

$$\frac{dy}{y} + \frac{dx}{cos^2x} = 0, \quad lny + tgx = lnC, \qquad y = Ce^{-tgx}$$

We are looking for a solution to the original inhomogeneous equation in the form

$$y = C(x)e^{-tgx},$$

where C(x) is an unknown function. Substituting the original equation

$$y = C(x)e^{-tgx}$$

and

$$y' = C'(x)e^{-tgx} - C(x)e^{-tgx}sec^2x$$

let's come to the equation

$$cos^2xC'e^{-tgx} - C(x)e^{-tgx}sec^2xcos^2x + C(x)e^{-tgx} = tgx$$

or

$$C'(x)cos^2(x)e^{-tgx} = tgx$$

where

$$C(x) = \int\frac{e^{tgx}tgx}{cos^2x}dx = e^{tgx}(tgx - 1) + C$$

We obtain a general solution to this equation:

$$y = tgx - 1 + Ce^{-tgx}$$

Using the initial condition $y(0) = 0$, we obtain $0 = -1 + C$, from where C = 1. Consequently, the required particular solution has the form

$$y = tgx - 1 + Ce^{-tgx} = tgx - 1 + 1 \cdot e^{-tgx}$$

$$y = tgx - 1 + e^{-tgx}. ■$$

Example 3. Integrate equation

$$y' + \frac{xy}{1 - x^2} = arcsinx + x$$

Solution. We integrate the corresponding homogeneous equation:

$$y' + \frac{xy}{1 - x^2} = 0; \quad \frac{dy}{y} = -\frac{xdx}{1 - x^2}$$

$$lny = \frac{1}{2}ln(1 - x^2) + lnC$$

that is $y = C\sqrt{1 - x^2}$.
We now believe

$$y = C(x)\sqrt{1 - x^2}; \qquad then$$

$$y' = C'(x)\sqrt{1-x^2} - \frac{xC(x)}{\sqrt{1-x^2}}$$

After substituting the initial inhomogeneous equation we obtain

$$y' = C'(x)\sqrt{1-x^2} - \frac{xC(x)}{\sqrt{1-x^2}} + \frac{x}{1-x^2}C(x)\sqrt{1-x^2} = arcsinx + x$$

that is

$$C'(x) = \frac{\text{arcsinx}}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$$

Integrating, we find

$$C(x) = \int \left[\frac{arcsinx}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right] dx = \frac{1}{2}(arcsinx)^2 - \sqrt{1-x^2} + C$$

The general solution to this equation has the form

$$y = \sqrt{1-x^2}[(arcsinx)^2 - \sqrt{1-x^2} + C].\ \blacksquare$$

Example 4. Solve the equation $y' + \frac{y}{x} = x^2y^4$.

Solution. This is the Bernoulli equation. Let's integrate it using the method of varying an arbitrary constant. To do this, we first integrate the corresponding linear homogeneous equation

$$y' + \frac{y}{x} = 0,$$

the decision of which $y = \frac{C}{x}$.

We are looking for a solution to the original Bernoulli equation, assuming

$$y = \frac{C(x)}{x}, \qquad y' = \frac{C'(x)}{x} - \frac{C(x)}{x^2}$$

Substituting y and y' into the original equation gives

$$\frac{C'(x)}{x} - \frac{C(x)}{x^2} + \frac{C(x)}{x^2} = x^2\left[\frac{C(x)}{x}\right]^4 \quad \text{or} \quad \frac{C'(x)}{x} = \frac{[C(x)]^4}{x^2}$$

Let's integrate the resulting equation:

$$\frac{dC(x)}{[C(x)]^4} = \frac{dx}{x}; \quad -\frac{1}{3[C(x)]^3} = lnx - lnC$$

$$C(x) = \frac{1}{\sqrt[3]{3ln\left(\frac{C}{x}\right)}}.$$

General solution to the original equation

$$y = \frac{C(x)}{x} = \frac{1}{x\sqrt[3]{3ln\left(\frac{C}{x}\right)}}. \quad ■$$

Example 5. Integrate equation

$$y' - \frac{2xy}{1+x^2} = 4\frac{\sqrt{y}}{\sqrt{1+x^2}} arctgx.$$

Solution. This is the Bernoulli equation. We integrate it using the Bernoulli method, for which we set $y = uv$. Substituting the original equation

$$y = uv, \qquad y' = u'v + uv'$$

Let's group the terms containing u in the first place:

$$u'v + u\left(v' - \frac{2xv}{1+x^2}\right) = 4\frac{\sqrt{uv}}{\sqrt{1+x^2}} arctgx$$

Let us take as v some particular solution of the equation

$$v' - \frac{2xv}{1+x^2} = 0.$$

Separating the non-variables, we find

$$\frac{dv}{v} = \frac{2xdx}{1+x^2}; \; lnv = \ln(1+x^2); \; v = 1+x^2$$

(we do not enter the integration constant).

To find u we have the equation

$$u'v = 4\frac{\sqrt{uv}}{\sqrt{1+x^2}} arctgx,$$

or(since $v = 1 + x^2$)

$$u' = \frac{4\sqrt{u}arctgx}{1 + x^2}.$$

$$u = (arctg^2x + C)^2 \quad and \quad y = uv = (1 + x^2)(arctg^2x + C)^2$$

general solution of the original equation.

Try to decide for yourself [3]

1. Solve the equation $y' + \frac{n}{x}y = \frac{a}{x^n};\ y(1) = 0$
2. Integrate the equation $y' + 2xy = xe^{-x^2}$
3. Integrate the equation $y = xy' + y' lny$
4. Integrate the equation $(x^2lny - x)y' = y$

Reply.1) $y = \frac{a(x-1)}{x^n}$ 2) $y = e^{-x^2}\ (\frac{x^2}{2} + C)$

3) $x = Cy - 1 - lny$ 4) $x = \frac{1}{lny+1-Cy}$

8 - §. Equations in total differentials

Definition. The equation [1]

$$M(x,y)\,dx + N(x,y)dy = 0 \qquad (1)$$

is called a total differential equation if $M(x,y)$ and $N(x,y)$ are continuous, differentiable functions for which the relation holds

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (2)$$

$\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ continuous in some region.

Integration of equations in total differentials. Let us prove that if the left side of equation (1) is a complete differential, then condition (2) is satisfied, and vice versa - if condition (2) is satisfied, the left side of equation (1) is a complete differential of some function u(x, y), that is, equation (1) has the form

$$du(x,y) = 0 \quad (3)$$

and therefore its general integral is $u(x,y) = C$.

Let us first assume that the left side of equation (1) is the total differential of some function $u\,(x,y)$, that is

$$M(x,y)\,dx + N\,(x,y)dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Then

$$M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y}$$

Differentiating the first relation by y, the second – by x we obtain

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}$$

Assuming continuity of second derivatives, we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

that is, equality (2) is a necessary condition for the left-hand side of equation (1) to be the total differential of some function $u(x,\ y)$. Let us show that this condition is also sufficient, that is, that if equality (2) is satisfied, the left-hand side of equation (1) is the complete differential of some function $u(\,x,y\,)$.

We find the relationships $\frac{\partial u}{\partial x} = \mathrm{M(x,y\,)}$

$$u = \int_{x_0}^{x} M(x,y)dx + \varphi(y),$$

where x_0 is the abscissa of any point from the region of existence of the solution.

When integrating, we consider them to be constant, so an arbitrary integration constant can depend on. Let us select the function $\varphi(x)$ so that the second of relations (4) is satisfied. To do this, we differentiate both sides of the last equality by y and equate the result to $N(x,y)$:

$$\frac{\partial u}{\partial y} = \int_{x_0}^{x} \frac{\partial M}{\partial y} dx + \varphi'(y) = N(x, \; y)$$

but since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

then we can write

$$\int_{x_0}^{x} \frac{\partial N}{\partial y} dx + \varphi'(y) = N,$$

that is.

$$N(x,y)|_{x_0}^{x} + \varphi'(y) = N(x,y) \quad \text{or} \quad N(x,y) - N(x0,y) + \varphi'(y) = N(x,y)$$

Hence

$$\varphi'(y) = N(x_0, y)$$

or

$$\varphi(y) = \int_{y_0}^{y} N(x_0 \;, y) dy + C_1$$

Thus, the function $u(x,y)$ will have the form

$$u = \int_{x_0}^{x} M(x,y) dx + \int_{y_0}^{y} N(x_0 \;, y) dy + C_1$$

Here $P(x_0, y_0)$ is a point in the vicinity of which there is a solution to differential equation (1).

Equating this expression to an arbitrary constant C, we obtain the general integral of equation (1):

$$\int_{x_0}^{x} M(x, y) dx + \int_{y_0}^{y} N(x_0 \;, y) dy = C \quad (5)$$

Example 1. Given equation $\frac{2x}{y^3} dx + \frac{y^2 - 3x^2}{y^4} dy = 0$

Let's check whether these are complete differential equations. Let's denote

$$M = \frac{2x}{y^3}, \quad N = \frac{y^2 - 3x^2}{y^4}$$

Then

$$\frac{\partial M}{\partial y} = -\frac{6x}{y^4}, \qquad \frac{\partial N}{\partial x} = -\frac{6x}{y^4}$$

Condition (2) is $y \neq 0$ satisfied. This means that the left side of this equation is the complete differential of some unknown function $u(x, y)$. Let's find this function. So then, therefore $\frac{\partial u}{\partial x} = \frac{2x}{y^3}$

$$u = \int \frac{2x}{y^3} dx + \varphi(y) = \frac{x^2}{y^3} + \varphi(y)$$

where $\varphi(y)$ - is a function that is not yet defined y.

Differentiating this relationship by y and taking into account that

$$\frac{\partial u}{\partial y} = N = \frac{y^2 - 3x^2}{y^4}$$

we find

$$-\frac{3x^2}{y^4} + \varphi'(y) = \frac{y^2 - 3x^2}{y^4}$$

hence,

$$\varphi'(y) = \frac{1}{y^2}, \qquad \varphi(y) = -\frac{1}{y} + C_1$$

$$u(x, y) = \frac{x^2}{y^3} - \frac{1}{y} + C_1$$

Thus, the general integral of the original equation is

$$\frac{x^2}{y^3} - \frac{1}{y} = C.$$

Example 2. The equation [9]

$$(3x^2 + 10xy)dx + (5x^2 - 1)dy = 0$$

will be an equation in complete differentials throughout the XOY plane, since the functions

$$M(x,y) = 3x^2 + 10xy, \quad N(x,y) = 5x^2 - 1$$

and their partial derivatives

$$\frac{\partial M(x,y)}{\partial y} = 10\text{x}, \qquad \frac{\partial N(x,y)}{\partial x} = 10x$$

everywhere continuous

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The equation

$$x^2 y dx - (5xy + 1)dy = 0$$

is not an equation in total differentials, since there is no region in the XOY plane in which the partial derivatives:

$$\frac{\partial M}{\partial y} = x^2 \quad and \quad \frac{\partial N}{\partial x} = -5y$$

were identically equal.

If the equation

$$M(x,y)dx + N(x,y)dy = 0$$

1. is an σ equation in total differentials in the domain, then it can be written in this domain as:

$$du(x,y) = 0$$

where $u(x,y)$ is some function.

Let $y = \varphi(x)$ be any solution of the equation lying in the domain σ. Then $du(x, \varphi(x)) \equiv 0$, $u(x,y) = C$, where C- is a certain number.

Conversely, for any function $y = \varphi(x)$ implicitly given by the equation $u(x,y) = C$, where C - is a number, we have $du(x,y) = 0$, $that\ is$, y = φ(x) is the solution to equation (1).

The relation $u(x,y) = C$, where C - is an arbitrary constant, is the general integral of equation (1) in the region σ.

Example 3. The equation [3]

$$[cos(x + y) + 2]dx + [cos(x + y) - 5]dy = 0$$

is an equation in complete differentials throughout the XOY plane. Because

$$[cos(x + y) + 2]dx + [cos(x + y) - 5]dy = d[sin(x + y) + 2x - 5y],$$

then this equation has the form:

$$d[sin(x + y) + 2x - 5y] = 0.$$

The general integral of this equation in the XOY plane is the expression

$$sin(x + y) + 2x - 5y = C$$

where C is an arbitrary constant.

Try to decide for yourself [3]

1. Find the general integral of the equation
$$(e^x + y + siny)dx + (e^y + x + xcosy)dy = 0$$
2. Find the general integral of the equation
$$(x + y - 1)dx + (e^y + x)dy = 0$$
3. Find the general integral of the equation
$$(x\cos y - ysiny)dy + (x\,siny + ycosy)dx = 0$$
4. Find the general integral of the equation
$$(x + siny)dx + (xcosy + siny)dy = 0$$
5. Find the general integral of the equation
$$(y + e^x siny)dx + (x + e^x cosy)dy = 0$$
6. Solve equations.
$$(x^2 + siny)dx + (1 + xcosy)dy = 0$$
7. $(xy + siny)dy + (0{,}5x^2 + xcosy)dy = 0$
8. $(x^2 + y^2 + y)dx + (2xy + x + ey)dy = 0;\ y(0) = 0.$
9. $(y + xlny)dx + (\frac{x^2}{2y} + x + 1)dy = 0$
10. $(3x^2y + sinx)dx + (x^2 + 1 + arctgy)dy = 0$

Answers. 1) $e^x + xy + xsiny + e^y = C$

2) $\frac{1}{2}x^2 + xy - x + e^y - 1 = C_1,\ e^y + \frac{1}{2}x^2 + xy - x = C,\ or\ C = C_1 + 1$

3) $u(x, y) = xe^x siny + e^x ycosy - e^x siny = C$ or

$e^x(xsiny + ycosy - siny) = C$

4) $\frac{1}{2}x^2 + 2xy - y^2 - 4x + 8y = C$

5) $xy + e^x siny = C$

6) $x^3 + 3y + 3xsiny = C$

7) $\frac{1}{2}x^2y + xsiny = C$

8) $\frac{1}{3}x^3 + xy^2 + xy + e^y = 1$

9) $x^2 lny + 2y(x + 1) = C$

10) $x^3y - cosx - siny = C$

9 - §. Integrating factor

Let the left side of the equation [1]

$$M(x,y)\,dx + N(x,y)dy = 0 \qquad (1)$$

there is no full differential. Sometimes it is possible to select a function such that μ (x, y), after multiplying all the terms of the equation, the left side of the equation becomes a complete differential. The general solution of the equation obtained in this way coincides with the general solution of the original equation; the function is called the integrating factor of equation (1).

In order to find the integrating factor, we proceed as follows: multiply both sides of this equation by the still unknown integrating factorμ:

$$\mu\, Mdx + \mu\, Ndy = 0$$

In order for the last equation to be an equation in total differentials, it is necessary and sufficient that the relation be satisfied

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

that is

$$\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$

or

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

After dividing both sides of the last equation by, we getμ

$$M\frac{\partial ln\mu}{\partial y} - N\frac{\partial ln\mu}{\partial x} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \qquad (2)$$

It is obvious that any function μ(x, y)that satisfies the last equation is an integrating factor of equation (1). Equation (2) is a partial differential equation with an unknown function μ, depending on two variables x and y. It can be proven that under certain conditions it has an infinite number of solutions and, therefore, equation (1) has an integrating factor. But in general, the problem of finding equation (2) is even more difficult than the original problem of integrating equation (1). Only in some special cases is it possible to find the function μ (x,y).

Let, for example, equation (1) admit an integrating factor that depends only on y. Then

$$\frac{\partial ln\mu}{\partial x} = 0$$

and to find we obtain μ the ordinary differential equation

$$\frac{\partial ln\mu}{\partial y} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

from which lnμ is determined, and μ therefore. It is clear that this can be done only in the case $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ if the expression does not depend on y, but depends only on x, then an integrating factor that depends only on x can easily be found.

Example 1. Solve the equation $(y + xy^2)dx - xdy = 0$

Solution. Here $M = y + xy^2$, $N = -x$

$$\frac{\partial M}{\partial y} = 1 + 2xy, \qquad \frac{\partial N}{\partial x} = -1, \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Therefore, the left side of the equation is not a complete differential. Let's see if this equation does not admit an integrating factor that depends only on. Noticing that

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-1 - 1 - 2xy}{y + xy^2} = -\frac{2}{y}$$

We conclude that the equation admits an integrating factor that depends only on. We find it:

$$\frac{d\ln\mu}{dy} = -\frac{2}{y}; \text{ from here } \quad \ln\mu = -2\ln y, \ \textit{that is } \mu = \frac{1}{y^2}$$

After multiplying all terms of this equation by the found integrating factor, we obtain the equation μ

$$\left(\frac{1}{y} + x\right) dx - \frac{x}{y^2} dy = 0$$

in full differentials

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\frac{1}{y^2}.$$

Solving this equation, we find its general integral

$$\frac{x}{y} + \frac{x^2}{2} + C = 0 \qquad \text{or} \quad y = -\frac{2x}{x^2+2C}. \quad \blacksquare$$

Example 2. Find the general integral of the equation

$$(x\cos y - y\sin y)dy + (x\sin y + y\cos y)dx = 0$$

Solution. We have

$$P(x, y) = x\sin y + y\cos y, \qquad Q(x, y) = x\cos y - y\sin y$$

$$\frac{\partial P}{\partial y} = x\cos y + \cos y - y\sin y, \qquad \frac{\partial Q}{\partial x} = \cos y$$

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{xcosy - ysiny}{xcosy - ysiny} = 1$$

Therefore, this equation has an integrating factor that depends only on x. Let's find this integrating factor:

$$\mu = e^{\int \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} dx} = e^{\int dx} = e^x$$

Multiplying the original equation by, we get the equation e^x

$$e^x(xcosy - ysiny)dy + e^x(xsiny + ycosy)dx = 0$$

which, as is easy to see, is already an equation in total differentials; in fact, we have

$$P_1(x,y) = e^x(xsiny + ycosy), \qquad Q_1(x,y) = e^x(xcosy - ysiny).$$

From here

$$\frac{\partial P_1}{\partial y} = \frac{\partial}{\partial y}[e^x(xsiny + ycosy)] = e^x(xcosy + cosy - ysiny)$$

$$\frac{\partial Q_1}{\partial x} = \frac{\partial}{\partial x}[e^x(xcosy - ysiny)] = e^x(xcosy - ysiny + cosy)$$

These derivatives are equal and the left part of the resulting equation has the form $du(x,y)$.

$$\frac{\partial u}{\partial y} = e^x(xcosy - ysiny), \qquad \frac{\partial u}{\partial x} = e^x(xsiny + ycosy)$$

Integrating the first of these equalities, we find

$$u = \int e^x\,(xcosy - ysiny)dy + C(x) = xe^x siny + e^x ycosy - e^x siny + C(x)$$

Let's find the derivative of the resulting function:

$$\frac{\partial u}{\partial x} = e^x siny + xe^x siny - e^x siny + e^x ycosy + C'(x) =$$

$$= e^x(xsiny + ycosy) + C'(x)$$

Comparing the found value $\frac{\partial}{\partial x}$ with $P(x,y)$, we obtain $C'(x) = 0$, that is, $C(x) = 0$. The general integral of the original equation has the form

$$u(x,y) = xe^x siny + e^x ycosy - e^x siny = C$$
$$or$$
$$e^x(xsiny + ycosy - siny) = C. \quad \blacksquare$$

Try to decide for yourself [3]

Integrate the following equations that have an integrating factor that depends only on or only on:

1) $ydx - xdy + lnxdx = 0$ $(\mu = \varphi(x))$.
2) $(x^2cosx - y)dx + xdy = 0$ $(\mu = (x))$.
3) $ydx - (x + y^2)dy = 0$; $(\mu = \varphi(y))$.
4) $y\sqrt{1 - y^2}dx + (x\sqrt{1 - y^2} + y)dy = 0$; $(\mu = \varphi(y))$.

Answers.

1) $y = Cx - lnx - 1$; $\mu = \frac{1}{x^2}$
2) $y = x(C - sinx)$; $\mu = \frac{1}{x^2}$
3) $x = y(C + y)$; $\mu = \frac{1}{y^2}$
4) $xy - \sqrt{1 - y^2} = C$; $\mu = \frac{1}{\sqrt{1-y^2}}$

10 - §. Riccati equations

$$y' + y^2 = Ax^\alpha$$

integrates squarely only for certain values, and in particular, the equation α

$$y' + y^2 = x$$

does not integrate in quadratures. [4]

First-order differential equation of the form

$$\frac{dy}{dx} + a(x)y^2 + b(x)y + c(x) = 0 \qquad (1)$$

where $a(x), b(x), c(x)$ are known functions.

(1) called the Riccati equation (generalized). If the coefficients a, b, c in the Riccati equation are constant, then the equation allows for the separation of variables and we immediately obtain a general integral

$$C_1 - x = \int \frac{dy}{ax^2 + by + c}$$

As Liouville showed, equation (1) is generally not integrable by quadratures.

Properties of the Riccati equation

1. If some particular solution to $y_1(x)$ equation (1) is known, then its general solution can be obtained using quadratures.

In fact, let's put

$$y = y_1(x) + z(x) \qquad (2)$$

where $z(x)$ is the new unknown function.

Substituting (2) into (1), we find

$$\frac{dy_1}{dx} + \frac{dz}{dx} + a(x)(y_1^2 + 2y_1 z + z^2) + b(x)(y_1 + z) + c(x) = 0$$

from where, due to the fact that $y_1(x)$ is a solution to equation (1), we obtain

$$\frac{dy}{dx} + a(x)(2y_1 z + z^2) + b(x)z = 0,$$

or

$$\frac{dy}{dx} + a(x)z^2 + [2a(x)y_1 + b(x)]z = 0 \qquad (3)$$

Equation (3) is a special case of the Bernoulli equation.

Example 1. Solve the Riccati equation

$$y' - y^2 + 2e^x y = e^{2x} + e^x \qquad (4)$$

knowing its particular solution $y_1 = e^x$.

Solution. Put we get $y = e^x + z(x)$ and substitute into equation (4);

$$\frac{dz}{dx} = z^2, \qquad where \ -\frac{1}{z} = x - C, \ or \ z = \frac{1}{C - x}$$

Thus, the general solution to equation (4)

$$y = e^x + \frac{1}{C-x}. \quad \blacksquare$$

Comment. Instead of substitution (2), it is often practically more profitable to substitute

$$y = y_1(x) + \frac{1}{u(x)}$$

which immediately leads the Riccati equation (1) to linear

$$u' - (2ay_1 + b) = a.$$

2. If two partial solutions of equation (1) are known, then its general integral is found by one quadrature.
Let two partial solutions of equation (1) be known. Using $y_1(x)$ *and* $y_2(x)$ the fact that the identity holds

$$\frac{dy_1}{dx} \equiv -a(x)y_1^2 - b(x)y_1 - c(x)$$

Let's represent equation (1) in the form

$$\frac{1}{y-y_1}\frac{d(y-y_1)}{dx} = -a(x)(y+y_1) - b(x)$$

or

$$\frac{d}{dx}[ln y - y_1] = -a(x)(y + y_1) - b(x) \qquad (5)$$

For the second particular $y_2(x)$ solution we similarly find

$$\frac{d}{dx}[ln y - y_2] = -a(x)(y + y_2) - b(x) \qquad (6)$$

Subtracting equality (5) equality (6), we get

$$\frac{d}{dx}[ln\frac{y-y_1}{y-y_2}] = a(x)(y_2 - y_1),$$

Where

$$\frac{y-y_1}{y-y_2} = C^{\int a(x)[y_2(x)-y_1(x)]dx} \qquad (7)$$

Example 2. The equation $\frac{dy}{dx} = \frac{m^2}{x^2} - y^2,\ m = const$ has partial solutions

$y_1 = \frac{1}{x} + \frac{m}{x^2},\ y_2 = \frac{1}{x} - \frac{m}{x^2}.$ Find its general integral.

Solution. Using formula (7), we obtain the general integral of the original equation

$$\frac{y-y_1}{y-y_2} = C^{\int \frac{-2m}{x^2}dx},$$

where

$$\frac{x^2y - x - m}{x^2y - x + m} = C^{\frac{2m}{x}}$$

11- §. Equations with individual variables.

Definition.A differential equation of the 1st order is the so-called equation with individual variables. This is an equation of the form [7]

$$f(x) + g(y)\,y' = 0 \qquad (1)$$

Replacing y' here and $\frac{d}{dx}$ multiplying the equation by dx, we find

$$f(x)dx + g(y)dy = 0 \qquad (2)$$

$f(x)dx-$ depends only on ty x , and the other $g(y)dy$ – only on y – the variables are separated. To solve equation (2), it is enough to integrate this equation, which gives

$$\int f(x)dx + \int g(y)dy = C$$

$$F(x) + G(y) = C \qquad (3)$$

If we solve equation (3) relative, we get the equality

$$y = \varphi(x, C) \qquad (4)$$

the right side of which is the general solution of equation (1).

Example 1. Apply the above to the differential equation

$$2x + \frac{y'}{y} = 0$$

Replacing y' on $\frac{d}{dx}$ and multiplying by dx, we get

$$2xdx + \frac{dy}{y} = 0$$

Integrating, we find

$$x^2 + lny = C$$

From here

$$lny = C - x^2 \quad or \quad y = e^{C-x^2}.$$

This equality can be rewritten as

$$y = e^C \cdot e^{-x^2}.$$

Let's denote e^C by C_1, and then again instead of C_1 we will use C. General solution of the differential equation in the form

$$y = Ce^{-x^2}.$$

This is called the general integral of a differential equation. General integral of the differential equation

$$F(x, y, y') = 0 \qquad (5)$$

this equation is called

$$F(x, y, C) = 0 \qquad (6)$$

x, y and C, solving which with respect to y, we find the general solution (5). Since finding u from (6) can present significant difficulties, but they are only of an algebraic nature. If the general integral or solution of the equation is expressed through non-elementary integrals, then it is considered found.

Example 2. a) Integrating the equation

$$2xdx + (5y^4 + cosy)dy = 0 \qquad (7)$$

we find its general integral

$$x^2 + y^5 + siny = C \qquad (7a)$$

Hence y, through x and C, equation (7a) is solved. b) for the equation

$$2xdx + e^{-y^2}dy = 0$$

general integral

$$x^2 + \int e^{-y^2}dy = C \qquad (7b)$$

solved the equation, although the integral $\int e^{-y^2} dy$ is not expressed through elementary functions.

At first glance, this approach seems like something like self-deception: after all, the solution has not been found. For equation (7), general integral (7a) and fixing some - or C in it. When the general integral of a differential equation can be written in a form containing indefinite integrals, the equation integrates in quadratures.

Every differential equation with separated variables is integrated by quadratures. There are differential equations for which this is not true.

12 - §.Claireau equation

Let's consider the so-called Clairaut equation [1]

$$y = x\frac{dy}{dx} + \psi\left(\frac{dy}{dx}\right) \qquad (1)$$

It is integrated by introducing an auxiliary parameter. Namely, let us then $\frac{dy}{dx}$ = p; put equation (1) in the form

$$y = xp + \psi(R) \qquad (1')$$

Let us differentiate all terms of the last equation from $x,$ keeping in mind that $p = \frac{d}{dx}$ is a function of x:

$$p = x\frac{dp}{dx} + p + \psi'(p)\frac{dp}{dx}$$

or

$$[x + \psi'(p)]\frac{dp}{dx} = 0$$

Equating each factor to zero, we get

$$\frac{dp}{dx} = 0 \qquad (2)$$

and

$$x + \psi'(p) = 0 \qquad (3)$$

1) Integrating equality (2), we obtain $p = C$ $(C = const)$. Substituting this value into equation (1′), we find its general integral

$$y = xC + \psi(C) \quad (4)$$

which from a geometric point of view represents a family of straight lines.

2) If we find m from equation (3) as a function and substitute it into equation (1′), then we obtain the function

$$y = xp(x) + \psi[p(x)] \qquad (1'')$$

which, as can be easily shown, is a solution to equation (1).

In fact, by virtue of equality (3) we find

$$\frac{dy}{dx} = p + [x + \psi'(p)]\frac{dp}{dx},$$

that is

$$\frac{dy}{dx} = p.$$

Therefore, substituting the function (1″) into equation (1), we obtain the identity

$$xp + \psi(p) = xp + \psi(R).$$

The solution (1″) cannot be obtained from the general integral (4) for any value of C. This is a special problem; it is obtained by eliminating the parameter from the equations

$$y = xp + \psi(R), \qquad x + \psi'(p) = 0$$

or, what is the same, with the exception of the equations

$$y = xC + \psi(C), \qquad x + \psi'_C(C) = 0$$

Consequently, a special solution of the Clairaut equation determines the envelope of the family of straight lines defined by the general integral (4).

Example 1. Find the general and special integrals of the equation

$$y = x\frac{dy}{dx} + \frac{a\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

Solution. We obtain the general integral by replacing $\frac{d}{dx}$ with C:

$$y = xC + \frac{aC}{\sqrt{1 + C^2}}$$

To obtain a special solution, differentiate the last equation with respect to C:

$$x + \frac{a}{(1 + C^2)^{3/2}} = 0$$

The special solution is obtained in parametric form

$$x = -\frac{a}{(1 + C^2)^{3/2}}, \quad y = \frac{aC^3}{(1 + C^2)^{3/2}}$$

By excluding parameter C, we can obtain a direct dependence between x and y. Raising both parts of each equation to a power $\frac{2}{3}$ and adding the resulting equations term by term, we find a contradictory solution in the following form:

$$x^{2/3} + y^{2/3} = a^{2/3}$$

This is an astroid.

Example 2. Integrate the equation $y = xy' - e^{y'}$

Solution. Let us put $y' = p$ and rewrite the equation in the form of $y = px - e^p$. Differentiable:

$$dy = pdx + xdp - e^p dp;$$

But $dy = pdx$,
therefore the last equation takes the form

$$xdp - e^p dp = 0 \quad \text{or} \quad (x - e^p)dp = 0$$

$$dp = 0\,,\ x - e^p = 0,\ x = e^p,\ dp = 0,\ p = C;$$

Substituting this value for p the equality $y = px - e^p$, we obtain the general solution of this equation:

$$y = Cx - e^C.$$

If we put

$$x = e^p,\ that\ y = pe^p - e^p = (p-1)e^p$$

we arrive at a special solution to the original equation

$$\begin{cases} x = e^p \\ y = (p-1)e^p \end{cases}$$

Excluding the parameter p (in this case $p = lnx$), we find a unique solution in explicit form:

$$y = x(lnx - 1)$$

Differentiating the special problem, we find

$$y' = lnx.$$

Equation of a tangent to an oblique integral curve at a point $M(x_0;\ y_0)$ [where $y_0 = x_0(lnx_0 - 1)$] will be written in the form

$$y - y_0 = y'_0(x - x_0) \quad or \quad y - x_0(lnx_0 - 1) = lnx_0(x - x_0),$$

which after simplification gives

$$y = xlnx_0 - x_0\,.$$

If you put it here

$$lnx_0 = C,$$

then the equation of the family of tangents of the skew integral curve will take the form

$$y = Cx - e^C. \qquad ■$$

Try to decide for yourself [3]

1. Integrate equation $y = x\left(\frac{1}{x} + y'\right) + y'$

2. $y = xy' + \sqrt{b^2 + a^2y'^2}$

3. $x = \frac{y}{y'} + \frac{1}{y'^2}$

4. $y = xy' + y' - y'^2$

5. $y = xy' + y'$

6. $y = xy' - \frac{1}{y'^2}$

Answers.1) general solution $y = Cx + C^2 + 1$;

specialdecision $\begin{cases} x = -2p \\ y = 1 - p^2 \end{cases}$ or $y = 1 - \frac{x^2}{4}$

2) common decision $y = Cx + \sqrt{b^2 + a^2C^2}$;

special decision $\begin{cases} x = -\frac{a^2p}{\sqrt{b^2 + a^2p^2}} \\ y = \frac{b^2}{\sqrt{b^2 + a^2p^2}} \end{cases}$ *or* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

3) Common decision $y = Cx - \frac{1}{C}$;

specialdecision $\begin{cases} x = -\frac{1}{p^2} \\ y = -\frac{2}{p} \end{cases}$ or $y^2 = -4x$

4) general decision or $y = Cx + C(1 - C)$; *specialdecision* $\begin{cases} x = 2p - 1 \\ y = p^2 \end{cases}$ $y = \frac{1}{4}(x + 1)^2$

5) common decision $y = Cx + C$

6) common decision $y = Cx - \frac{1}{C^2}$; *specialdecision* $y^3 = -\frac{27}{4}x^2$

13 - §. Lagrange equation

Definition. Lagrange Equation called an equation of the form [1]

$$y = x\,\varphi(y') + \psi(y') \qquad (1)$$

where φ and ψ- known functions from $\frac{dy}{dx}$.

This equation is linear with respect to them. The Clairaut equation considered in the equation is a special case of the Lagrange equation at φ(y′). Integration of the Lagrange equation, as well as integration of the Clairaut equation, is carried out by introducing an auxiliary parameter. Let's put $\varphi(y') \equiv y'$.

$$y' = p;$$

then the original equation will be written in the form

$$y = x\varphi(p) + \psi(p) \qquad (1')$$

Differentiating from x, we get

$$p = \varphi(p) + [x\varphi'(p) + \psi'(R)]\ \frac{dp}{dx}$$

or

$$p - \varphi(p) = [x\varphi'(p) + \psi'(p)]\frac{dp}{dx} \qquad (1'')$$

From this equation it is reasonable to find some solutions: namely, it turns into an identity for any constant value $p = p_0$ satisfying the condition

$$p_0 - \varphi(p_0) = 0$$

Indeed, at a constant value of $p\frac{dp}{dx} \equiv 0$, the derivative and both sides of equation (1″) vanish.

The solution corresponding to each value $p = p_0$, that is, $\frac{dp}{dx} = p_0$, is a linear function of otx. In order to find this function, it is enough to substitute the value p = p_0 into equality (1′):

$$y = x\varphi(p_0) + \psi(p_0)$$

If it turns out that this solution cannot be obtained from a general solution for any value of an arbitrary constant, then it will be a special solution.

Let us now find a general solution. To do this, we write equation (1″) in the form

$$\frac{dx}{dp} - x\,\frac{\varphi'(p)}{p - \varphi(p)} = \frac{\psi'(p)}{p - \varphi(p)}$$

and we will consider$x$as a function of neg. Then the resulting equation will be a linear differential equation with respect to the function x top.

By solving it, we will find it

$$x = \omega(p, C) \qquad (2)$$

Eliminating the parameters of equations (1′) and (2), we obtain the general integral of equation (1) in the form

$$F(x, y, C) = 0$$

Example. Given equation

$$y = xy'^2 + y'^2 \qquad \text{(I)}$$

Let's put $y' = p$, we will have

$$y = xp^2 + p^2 \qquad \text{(I′)}$$

differentiating from x, we get

$$\text{p} = p^2 + [2xp + 2p] \frac{\phi}{dx} \qquad \text{(I″)}$$

We will find special solutions. Since $p = p^2$ at $p_0 = 0$ and $p_1 = 1$, then the solutions will be linear functions $y = x^2{}_0 + 0^2$, that is, $y = 0$ and $y = x + 1$.

Will these functions be particular or special solutions when we find the general integral? To search for it, we write the equation (I″) in the form

$$\frac{dx}{dp} - x\frac{2}{1-p} = \frac{2}{1-p}$$

and we will consider x as a function of the independent variable p. Integrating the resulting linear equation, we find

$$x = -1 + \frac{C^2}{(p-1)^2} \qquad \text{(II)}$$

excluding equations (I′) and (II), we obtain the general integral

$$y = (C + \sqrt{x+1})^2$$

The singular integral of the original equation will be $y = 0$, since this solution cannot be obtained from the general one at any value of C. The function $y = x + 1$ is not a special, but a particular solution; it is obtained from the general solution at $C = 0$.

Try to decide for yourself [3]

1. Integrate the equation $y = x y'^2 + y'^2$

2. $y = 2xy' + y'^2$

3. $2y(y' + 1) = xy'^2$

4. $y = x(1 + y') + (y')^2$

Answers.

1) $(\sqrt{y} + \sqrt{x+1})^2 = C$

2) 2) $x = \frac{C}{3p^2} - \frac{2}{3}p, \quad y = \frac{2C - p^3}{3p}$

3) common decision $\begin{cases} x = C(p+1) \\ y = \dfrac{Cp^2}{2} \end{cases}$ $or \quad y = \frac{(x-C)^2}{2C};$

$special decision \quad y = 0, \qquad y = -2x$

4) Common decision $x = Ce^{-p} - 2p + 2, \; y = C(p+1)e^{-p} - p^2 + 2$

14 -§. First-order linear equations (variation method)

The differential equation $y' = f(x, y)$ is called linear if it is linear with respect to the desired function and its derivative y', i.e. if it can be written as: [1]

$$y' + P(x)y = Q(x) \qquad (1)$$

Examples of linear equations:

$$y' + x^2 y = x^5, \qquad y' + x + e^x y = 0, \; y' = y \qquad etc.$$

Definition. If in equation (1) the right-hand side $Q(x)$ is not equal to zero, then this equation is called a linear inhomogeneous equation, or a linear equation with a right-hand side. If $Q(x) \equiv 0$, then equation (1) is called a linear homogeneous equation, or an equation without a right-hand side.

The equation $y' + P(x)y = 0$, obtained from equation (1) by replacing the function $Q(x)$ by zero, is called a linear homogeneous equation corresponding to this inhomogeneous equation.

We will consider the linear equation

$$y' + P(x)y = Q(x)$$

on the interval $a < x < b$ continuity of the functions $P(x)$ *and* $Q(x)$.

Let us show that such an equation can be integrated in quadratures.

Let us first take the linear homogeneous equation

$$y' + P(x)y = 0 \qquad (2)$$

corresponding to a given heterogeneous one.

This is an equation with separable variables. Separating the variables and integrating, we find:

$$\frac{dy}{y} = -P(x)dx,$$

$$ln|y| = -\int P(x)dx + ln|C|,$$

$$y = Ce^{-\int P(x)dx}, \text{ where}$$

C is an arbitrary constant, different from zero.

The solution y = 0, lost when separating the variables, is obtained from the relation

$$y = Ce^{-\int P(x)dx} \quad \text{at} \quad C = 0.$$

Ratio

$$y = Ce^{-\int P(x)dx} \qquad (3)$$

where C is an arbitrary constant and is a general solution to equation (2) in the strip $\{a < x < b, -\infty < y < +\infty\}$.

To find solutions to linear inhomogeneous equation (1), we apply the method of variation of an arbitrary constant. We will look for a solution to equation (1) in the same form (3) as the solution to the corresponding homogeneous equation. Then C will have to be considered non-constant, a function of $x, C = C(x)$. This function $C(x)$ must be such that upon substitution

$$y = Ce^{-\int P(x)dx} \quad \text{and} \quad y' = C'(x)e^{-\int P(x)dx} - C(x)P(x)e^{-\int P(x)dx}$$

in equation (1), it turned into the identity on the interval $a < x < b$.

To determine the function C(x), we obtain an equation with separable variables:

$$C'(x)e^{-\int P(x)dx} = Q(x).$$

Integrating it, we find:

$$C(x) = \int Q(x)e^{\int P(x)dx}\, dx + C$$

where C is an arbitrary constant.

For any value of constant C function

$$y = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + C\right) \qquad (4)$$

is a solution to equation (1).

On the contrary, since the function $e^{-\int P(x)dx}$ is nonzero, any solution to equation (1) in the domain $\{a < x < b, -\infty < y < +\infty\}$ can be written as:

$$y = C(x)e^{-\int P(x)dx}$$

which means, in the form of (4) at some value of the constant.
Relationship (4) is a general solution to equation (1) in the domain)
$\{a < x < b, -\infty < y < +\infty\}$. General solution of linear inhomogeneous equation (1)

$$y = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + C\right)$$

turns out to be equal to the sum of the general solution of the corresponding homogeneous equation (C) and the particular solution of this inhomogeneous equation

$$\left(e^{-\int P(x)dx}e^{-\int P(x)dx} \cdot \int Q(x)e^{\int P(x)dx}dx\right).$$

Example 1. $y' - 2xy = (x + y)e^{x^2}$ - linear inhomogeneous equation. The functions $P(x) = -2x$ and $Q(x) = (x + 1)e^{x^2}$ are continuous everywhere.We first solve the linear homogeneous equation $y' - 2xy = 0$, corresponding to this equation:

$$\frac{dy}{y} = 2xdx, \qquad ln|y| = x^2 + ln|C|, \qquad y = Ce^{x^2}$$

We are looking for a general solution to this equation in the form:

$$y = C(x)e^{x^2}$$

Then

$$y' = C'(x)e^{x^2} + C(x)2xe^{x^2}$$

Substituting y and y' the equation

$$y' - 2xy = (x + 1)e^{x^2}$$

after bringing similar terms, we get:

$$C'(x)e^{x^2} = (x + 1)e^{x^2}$$

where

$$C'(x) = x + 1, C(x) = \frac{(x+1)^2}{2} + C,$$

where C is an arbitrary constant.

The general solution to this equation in the entire XOY plane has the form:

$$y = [\frac{(x+1)^2}{2} + C]\, e^{x^2}. \qquad \blacksquare$$

***Solution of linear equation*(1).** We will look for a solution to equation (1) in the form of the product of two functions fromx:

$$y = u(x)v(x) \qquad (5)$$

One of these functions can be taken arbitrary, the other can be determined based on equation (1).

Differentiating both sides of equality (2), we find

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Putting the resulting derivative expression into $\frac{d}{dx}$ equation (1), we will have

$$u\frac{dv}{dx} + v\frac{du}{dx} + Puv = Q,$$

or

$$u\left(\frac{dv}{dx} + Pv\right) + v\frac{du}{dx} = Q \qquad (6)$$

Let us choose a function v such that

$$\frac{dv}{dx} + Pv = 0 \qquad (7)$$

Separating the variables in this differential equation with respect to the function v, we find

$$\frac{dv}{v} = -Pdx.$$

Integrating, we get

$$-ln|C_1| + ln|v| = -\int Pdx,$$

or

$$v = C_1 e^{-\int Pdx}$$

It is enough for us to have some nonzero solution to equation (7), then we take the function $v(x)$

$$v(x) = e^{-\int Pdx} (8)$$

Some $\int Pdx$ - where some kind of primitive. It is $v(x) \neq 0$ obvious that. Putting the found value into $v(x)$equation (6), we get (considering , that $\frac{dv}{dx} + Pv = 0$)

$$v(x)\frac{du}{dx} = Q(x),$$

or

$$\frac{du}{dx} = \frac{Q(x)}{v(x)},$$

where

$$u = \int \frac{Q(x)}{v(x)} dx + C.$$

Substituting u and v into formula (5), we finally get

$$y = v(x) \left[\int \frac{Q(x)}{v(x)} dx + C\right],$$

or

$$y = v(x) \int \frac{Q(x)}{v(x)} dx + Cv(x) \quad (9)$$

Example 2. Solve the equation

$$\frac{dy}{dx} - \frac{2}{x+1} y = (x+1)^3$$

Solution. We believe then $y = uv$,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Putting the $\frac{d}{dx}$ expression in the original equation, we will have

$$u\frac{dv}{dx} + v\frac{du}{dx} - \frac{2}{x+1} uv = (x+1)^3,$$

$$u\left(\frac{dv}{dx} - \frac{2}{x+1} v\right) + v\frac{du}{dx} = (x+1)^3. \qquad (10)$$

To determine v we obtain the equation

$$\frac{dv}{dx} - \frac{2}{x+1} v = 0 \text{ those } \frac{dv}{v} = \frac{2dx}{x+1}$$

where

$$ln|v| = 2ln|x+1|, \quad \text{or} \quad v = (x+1)^2$$

Substituting the expression of the function v into equation (10), we obtain the equation for determining u

$$(x+1)^2 \frac{du}{dx} = (x+1)^3 \quad \text{or} \quad \frac{du}{dx} = x+1$$

where

$$u = \frac{(x+1)^2}{2} + C.$$

The general integral of the given equation will have the form

$$y = \frac{(x+1)^4}{2} + C(x+1)^2 . \qquad ■$$

Example 3. Find a general solution to the equation $y' + 3y = e^{2x}$.

Solution. This equation is linear. Here $P(x) = 3$, $Q(x) = e^{2x}$. First we solve the corresponding homogeneous equation $y' + 3y = 0$. Separating the variables and integrating, we find

$$\frac{dy}{y} = -3dx$$

$$ln|y| = -3x + ln|C_1| \quad \text{or} \quad y = \pm C_1e^{-3x} = Ce^{-3x}$$

We are looking for a general solution to this equation in the form

$$y = C(x)e^{-3x}$$

Differentiating, we have

$$y' = C'(x)e^{-3x} - 3C(x)e^{-3x}.$$

Substituting the expressions for y and y′ into this equation, we obtain

$$C'(x)e^{-3x} = e^{2x}, \qquad C'(x) = e^{5x} \qquad or\ dC = e^{5x}dx$$

where

$$C(x) = \frac{1}{5}e^{5x} + C_2$$

where C_2 − is an arbitrary constant. The general solution to this equation has the form

$$y = C(x)e^{-3x} = \left(\frac{1}{5}e^{5x} + C_2\right)e^{-3x} \quad or \quad y = \frac{1}{5}e^{2x} + C_2e^{-3x}. \ \blacksquare$$

Try to decide for yourself [3]

1. Solve equations. $xy' + y = e^{-x}$

2. Solve equations. $y' = \frac{y}{x} + cos\frac{y}{x}$

3. Solve equations. $y' = 4 + \frac{y}{x} + \left(\frac{y}{x}\right)^2;\ y(1) = 2.$

4. Solve equations. $(x^4 + 6x^2y^2 + y^4)dx + 4xy(x^2 + y^2)dy = 0;\ y(1) = 0$

5. Solve equations. $3y\sin\left(\frac{3x}{y}\right)dx + \left[y - 3x\sin\left(\frac{3x}{y}\right)\right]dy = 0$

Answers.

1) $y = \frac{1}{x}(-e^{-x} + C) = -\frac{e^{-x}}{x} + \frac{C}{x}$

2) $1 + \sin\frac{y}{x} = Cx\cos\frac{y}{x}$

3) $arctg\frac{0{,}5y}{x} - 2lnn|x| = \frac{\pi}{4}$

4) $x^5 + 10x^3y^2 + 5xy^4 = 1$

5) $ln|y| - \cos\frac{3x}{y} = C$

15 - §. Equations reducible to linear

When integrating some first-order differential equations, the following remark can be used. [7]

If the function $y = \varphi(x, C)$ outside some region is a general solution of the equation

$$y' = f(x, y),$$

then the expression is the general integral of the differential equation

$$v(y) = \varphi(x, C)$$

$$v'(y)y' = f(x, v(y)) \qquad (1)$$

in the same area.

By substituting into equation $z = v(y)$ (1).

If the $y' = f(x, y)$ – linear differential equation is:

$$y' + P(x)y = Q(x)$$

a is$v(y)$any differentiable function *y*, then the general integral of the equation

$$v'(y)y' + P(x)v(y) = Q(x) \qquad (2)$$

outside of which area will be written as, $v(y) = \varphi(x, C)$

where $y = \varphi(x, C)$ is the general solution of equation (1) in the same region.

Solving any equation

$$v'(y)y' + P(x)v(y) = Q(x)$$

using substitution it is reduced $v(y) = z$ to the solution of a linear equation.

For example, the Bernoulli equation, that is, the equation

$$y' + P(x)y = Q(x)y^{\alpha} \qquad (3)$$

where α- is any real number, different from zero and one, reduced to linear after preliminary division of both parts of the equation by α and subsequent substitution $z = y^{1-\alpha}$.

If

$$z = y^{1-\alpha}, \quad \text{that} \quad z' = (1-\alpha)\, y^{-\alpha} y'.$$

Substituting z and z′ into the equation

$$y^{-\alpha}y' + P(x)y^{1-\alpha} = Q(x)$$

obtained by dividing by y^{α}, we have:

$$\frac{1}{1-\alpha}z' + P(x)z = Q(x)$$

or, also,

$$z' + (1-\alpha)P(x)z = (1-\alpha)Q(x)$$

This is a linear equation in z. If $P(x)$ and $Q(x)$ are continuous, it integrates by quadrature. Finding its general solution and $z = y^{1-\alpha}$ substituting, we obtain a set of solutions to the Bernoulli equation.

Example. Bernoulli Equation

$$y' + xy = x^3y^3 \quad (\alpha = 3)$$

after substitution it reduces to the linear equation $y^{-2} = z$

$$z' - 2xz = -2x^3$$

Solving it, we find

$$z = x^2 + 1 + Ce^{x^2}$$

where C is an arbitrary constant. The set of solutions to this equation:

$$y^{-2} = x^2 + 1 + Ce^{x^2} \text{ and } y = 0. \ \blacksquare$$

16 - §. Linear equations of the first order

Definition: First order linear equationis an equation that is linear with respect to an unknown function and its derivative. It looks like [9]

$$\frac{dy}{dx} + P(x)y = Q(x) \quad (1)$$

where $P(x)$ and $Q(x)$ - are given continuous functions of otx (or constants).

Solution of linear equation (1). We will seek a solution to equation (1) in the form of a product of two functions otx:

$$y = u(x)v(x) \qquad (2)$$

One of these functions can be taken arbitrary, the other will be determined based on equation (1).

Differentiating both sides of equality (2), we find

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Substituting the resulting derivative expression into $\frac{d}{dx}$ equation (1), we will have

$$u\frac{dv}{dx} + v\frac{du}{dx} + Puv = Q$$

or

$$u\left(\frac{dv}{dx} + Pv\right) + v\frac{du}{dx} = Q \qquad (3)$$

let us choose a function v such that

$$\frac{dv}{dx} + Pv = 0 \qquad (4)$$

Separating the variables in this differential equation with respect to the function$v$, we find

$$\frac{dv}{v} = -Pdx$$

Integrating, we get

$$-ln = -|C_1| + ln|v| \int Pdx$$

or

$$v = C_1 e^{-\int Pdx}$$

Since we only need some nonzero solution to equation (4), then the function

$$v(x) = e^{-\int Pdx} \qquad (5)$$

some $\int Pdx$ - where some kind of primitive. Obviously, $v(x) \neq 0$. Substituting the found value of v(x) into equation (3), we obtain (taking into account that $\frac{dv}{dx} + Pv = 0$)

$$v(x)\frac{du}{dx} = Q(x) \text{ or } \frac{du}{dx} = \frac{Q(x)}{v(x)}$$

where

$$u = \int \frac{Q(x)}{v(x)} dx + C$$

Substituting u and v into formula (2), we finally get

$$y = v(x) \left[\int \frac{Q(x)}{v(x)} dx + C\right]$$

or

$$y = v(x) \int \frac{Q(x)}{v(x)} dx + C\, v(x) \qquad (6)$$

Example. Solve the equation

$$\frac{dy}{dx} - \frac{2}{x+1} y = (x+1)^3$$

Solution. We assume $y = uv$, then

$$\frac{dy}{dx} = u \frac{dv}{dx} + \frac{du}{dx} v$$

Substituting the expression $\frac{d}{dx}$ into the original equation, we will have

$$u \frac{dv}{dx} + \frac{du}{dx} v - \frac{2}{x+1} uv = (x+1)^3$$

$$u\left(\frac{dv}{dx} - \frac{2}{x+1} v\right) + \frac{du}{dx} v = (x+1)^3$$

To determine v we obtain the equation

$$\frac{dv}{dx} - \frac{2}{x+1} v = 0, \text{ that is } \frac{dv}{v} = \frac{2dx}{x+1},$$

where

$$ln|v| = 2ln|x+1| \; or \; v = (x+1)^2.$$

Substituting the expression of the function *v* into equation (7), we obtain the equation for determining u

$$(x+1)^2\frac{du}{dx} = (x+1)^3 \text{ or } \frac{du}{dx} = x+1,$$

where

$$u = \frac{(x+1)^2}{2} + C .$$

Consequently, the general integral of the given equation will have the form

$$y = \frac{(x+1)^4}{2} + C(x+1)^2$$

Linear equations with constant coefficients are common in these applications.

$$\frac{dy}{dx} + ay = b \qquad (8)$$

Where a and b - are constants.

If you can solve using substitution (2) or by separating variables:

$$dy = (-ay + b)dx,$$

$$\frac{dy}{-ay+b} = dx,$$

$$-\frac{1}{a} ln|-ay+b| = x + C_1$$

$$\ln|-ay+b| = -(ax + C^*)$$

where $C^* = aC_1$, $-ay + b = e^{-(ax+C^*)}$,

$$y = -\frac{1}{a} e^{-(ax+C^*)} + \frac{b}{a},$$

orfinally

$$y = Ce^{-ax} + \frac{b}{a}$$

where denoted by $-\frac{1}{a}e^{-C^*} = C$. This is the general solution to equation (8).

Try to decide for yourself [3]

1. Integrate the equation $y'cos^2x + y = tgx$ with the initial condition $y(0) = 0.$

2. Integrate the equation. $y' + \frac{xy}{1-x^2} = arcsinx + x$

3. Solve the equation $xy' - y = x^2cosx$

Answers.

1) $y = tgx - 1 + e^{-tgx}$

2) $y = chx(shx + C)$

3) $y = \sqrt{1 - x^2}[\frac{1}{2}(arcsinx)^2 - \sqrt{1 - x^2} + C]$

17 - §. Special solutions of a first order differential equation

Let the differential equation

$$F(x, y, \frac{dy}{dx}) = 0 \quad (1)$$

has a common integral

$$\Phi(x, y, C) = 0 \quad (2)$$

Let us assume that the family of integral curves corresponding to equation (2) has an envelope. Let us prove that this envelope is also an integral curve of the differential equation (1). [1]

Indeed, at each of its points the envelope touches some curve of the family, that is, there is a common tangent. Consequently, at each common point the envelope and curve of the family have the same values of the quantities x, y, y'.

But for a curve from the family, the numbers x, y, y' satisfy equation (1). Therefore, the same equation is satisfied by the abscissa, ordinate, and slope of each point of the envelope. But this means that the envelope is a solution to this differential equation.

Since the envelope is not a family curve, its equation cannot be obtained from the general integral (2) for any particular value of C. A solution to a differential equation that is not obtained from a general integral and at what value of C and has as its graph the envelope of the family of integral curves included in the general solution is called a special solution to the differential equation.

Let the general integral be known

$$F(x, y, C) = 0$$

excluding C from this equation and Eq.

$$\Phi_C'(x, y, C) = 0,$$

we get the equation

$$\psi(x, y) = 0.$$

If this function satisfies a differential equation, then it is a special integral.

Note that at least two integral curves pass through each point of the curve representing a special solution, that is, at each point of the special solution the uniqueness of the solution is violated.

Note that the point at which the uniqueness of the solution to the differential equation is violated, that is, the point through which at least two integral curves pass, is called a singular point. Thus, a special solution consists of special points.

Example. Find a special solution to the equation

$$y^2(1 - y'^2) = R^2$$

Solution. Let's find its general integral. Let us resolve the equation relative to y':

$$\frac{dy}{dx} = \pm \frac{\sqrt{R^2 - y^2}}{y}$$

Separating the variables, we get

$$\frac{ydy}{\pm\sqrt{R^2 - y^2}} = dx$$

From here, integrating, we find the general integral

$$(x - C)^2 + y^2 = R^2$$

It is easy to see that the family of integral lines is a family of circles of radius R with the center of the abscissa. The envelope of the family of curves will be a pair of straight lines $y = \pm R$.

The functions $y = \pm R$ satisfy the differential equation. Therefore, this is a special integral.

18 - §. First-order equation, unresolved with respect to the derivative

First order equations not resolved with respect to the derivative, that is, equations of the form:

$$F(x, y, y') = 0 \qquad (1)$$

The equation $F(x,\ y,\ y') = 0$ implicitly specifies at each point $(x,\ y)$ of some region of the XOY plane one or more real values of y′. If at each such point we construct segments with angular coefficients equal to the value at this point, we obtain the so-called direction field, defined by the equation $F(x, y, y') = 0$. [9]

To integrate equation (1) means to find all its solutions, either explicitly or implicitly. Geometrically, this means finding all the curves whose tangent at each point coincides with one of the field directions at that point.

Suppose that outside some region D of the XOY plane, the equation

$F(x, y, y') = 0$ implicitly specifies m different real values of y':

$$y' = f_1(x, y),\; y' = f_2(x, y), \dots, y' = f_m(x, y) \qquad (2)$$

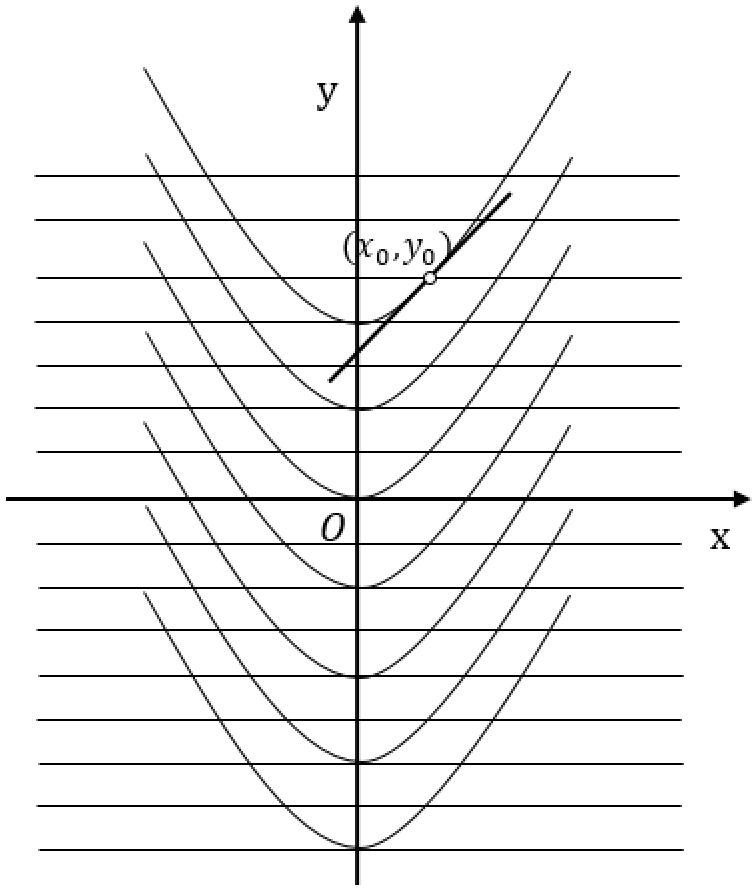
In this case, the direction field of the equation $F(x, y, y') = 0$ in the region D can be considered as a superposition of the m fields of equations (2) resolved with respect to the derivative. All solutions to these equations are solutions to this equation in region D.

Let $\Phi_1(x, y, C) = 0,\; \Phi_2(x, y, C) = 0, \dots, \Phi_m(x, y, C) = 0$ – general integrals of equations (2) in the domain D^2. The set of these common integrals is called the common integral of the equation $F(x, y, y') = 0$ in the domain D.

Example 1. The equation $y'^2 - 2xy' = 0$ is not resolved with respect to the derivative. Resolving it, we obtain $y' = 0$ and $y' = 2x$. General solutions of these equations throughout the XOY plane have the form:
$y = C$ and $y = x^2 + C$ (C - is an arbitrary constant). Therefore, the general integral of this equation in the XOY plane is given by two relations:

$$y = C, \qquad y = x^2 + C.$$

The direction field defined by this equation is obtained by superimposing the fields of the equations $y' = 0$ and $y' = 2x$. Through each point (x_0, y_0) of the XOY plane there pass two integral curves - a straight line
$y = y_0$ and parabola $y = x^2 + C_0$, where $C_0 = y_0 - x^2{}_0$. (Fig. 1)



Example 2. The equation $y'^3 - 1 = 0$ is not resolved with respect to the derivative. Resolving it relative to y', we obtain three values:

$$y' = 1, \qquad y' = \frac{-1 + \sqrt{3}i}{2}, \qquad y' = \frac{-1 - \sqrt{3}i}{2}$$

- one real and two imaginary.

Since we are only interested in real solutions of the equation, we consider only the equation $y' = 1$. Its general solution over the entire XOY plane has the form:

$$y = x + C,$$

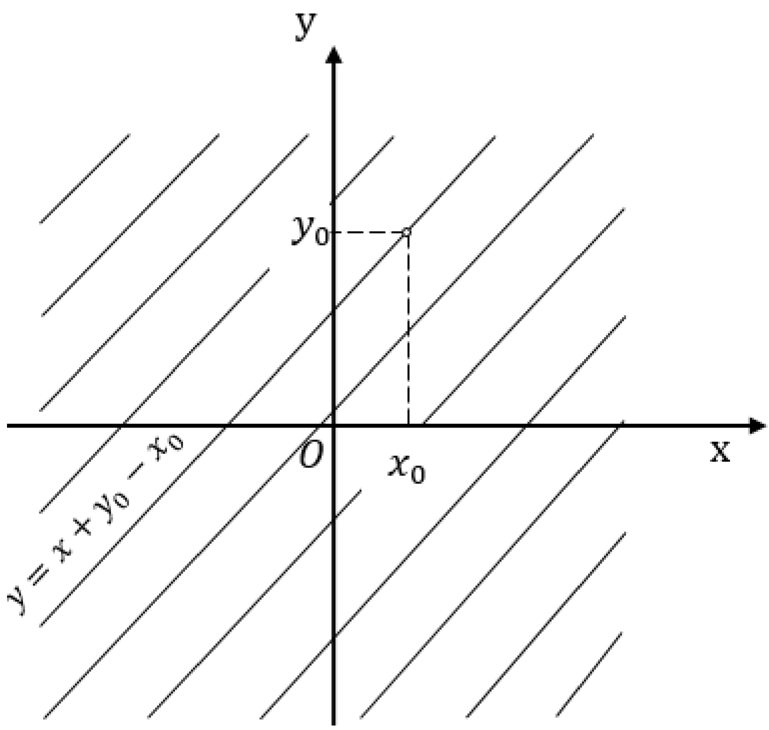
where C is an arbitrary constant. All (real) solutions of this equation are covered by the relation

$$y = x + C.$$

Through each point (x_0, y_0) of the XOY plane one integral curve of this equation passes:

$$y = x + C_0,$$

where $C_0 = y_0 - x_0$. (Fig. 2)



Example 3. The equation $y'^3 - 4yy' = 0$ defines at each point of the upper half-plane $(y > 0)$ three real values of y':

$$y' = 0, \qquad y' = 2\sqrt{y}, \quad y' = -2\sqrt{y}$$

Solving these equations, we obtain their general integrals in the upper half-plane:

$$y = C, \quad \sqrt{y} = x + C \ (x + C > 0), \qquad \sqrt{y} = -x + C \ (-x + C > 0),$$

where C- is an arbitrary constant.

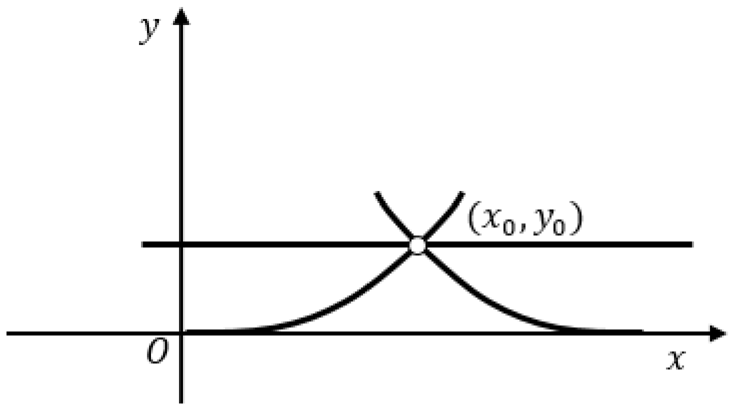
In the upper half-plane, the general integral of this equation is given by the relations:

$$y = C, \qquad \sqrt{y} = x + C \ (x + C > 0), \qquad \sqrt{y} = -x + C \ (-x + C > 0).$$

Three integral curves of this equation pass through each point (x_0, y_0) of the upper half-plane:

$$y = y_0, \qquad \sqrt{y} = x - x_0 + \sqrt{y_0}, \qquad \sqrt{y} = -x + x_0 + \sqrt{y_0}$$

(straight line and "branches" of two parabolas) (Fig. 3)



In the lower half-plane $(y < 0)$ the equation $y'^3 - 4yy' = 0$ has only one real solution relative to y':

$$y' = 0.$$

The general solution to the equation $y' = 0$ in the lower half-plane has the form:

$$y = C.$$

In the lower half-plane, the general integral of this equation has the form

$$y = C$$

Through each point (x_0, y_0) of the lower half- plane there passes one integral curve - straight line $y = y_0$.

19 - §. An equation that does not contain an explicit function

An equation that does not contain the explicitly sought function has the form:

$$F(x, y') = 0 \qquad (1)$$

If this equation can be represented as:

$$x = f(y') \qquad (2)$$

then all solutions to this equation can be found using a single square in parametric form. [9]

Indeed, we will consider the derivative of y' as a parameter: $y' = t$. Let $y = \varphi(x)$ be any solution of equation (2). The expression x through the parameter t for this solution is given by the equation itself:

$$x = f(t).$$

Let's find a parametric expression for y.
Because

$$y' = \frac{dy}{dx}, \text{ that } dy = y'dx$$

Replacing y' and dx with expressions through t:

$$y' = t, \qquad dx = f'(t)dt$$

we get:

$$dy = tf'(t)dt$$

whence, integrating, we find:

$$y = \int tf'(t)dt + C$$

where C - is a number.

Any solution to equation (2) can be written in parametric form as:

$$\begin{cases} x = f(t) \\ y = \int t f'(t)dt + C \end{cases} \quad (3)$$

where C - is a number.

If we exclude *t* from relations (3), we obtain (in explicit or implicit form) the solution $y = \varphi(x)$.

By direct substitution into equation (2), we make sure that for any value of the constant C, relations (3) determine the solution to this equation.

Relation (3), where C is an arbitrary constant, covers all solutions of equation (2).

Example 1. The equation $x = y' + e^{y'}$ cannot be resolved relative to y' in elementary functions, but it is already resolved relative to x. Let us find its solutions in parametric form.

We accept $y' = t$ parameter. Then

$$x = t + e^t, \qquad dx = (1 + e^t)dt, \qquad dy = y'dx, \qquad dy = t(1 + e^t)dt$$

$$y = \int t(1 + e^t)dt + C = \frac{t^2}{2} + te^t - e^t + C$$

A set of solutions to this equation in parametric form

$$\begin{cases} x = t + e^t \\ y = \dfrac{t^2}{2} + e^t(t - 1) + C \end{cases}$$

Example 2. The equation $y'^2 = x$ can also be solved by introducing a parameter. Let

$$y' = t, \qquad then \;\; x = t^2, \qquad dx = 2tdt, \qquad dy = y'dx,$$

that is

$$dy = 2t^2dt$$

which means

$$y = \frac{2}{3}t^3 + C,$$

where C- is any number.

The set of solutions to the equation $y'^2 = x$ in parametric form has the form:

$$\begin{cases} x = t^2 \\ y = \frac{2}{3}t^3 + C \end{cases}$$

Excluding the parameter t, it is easy to obtain solutions to this equation in the form:

$$y = \pm\frac{2}{3}x^{\frac{3}{2}} + C.\blacksquare$$

Try to decide for yourself [3]

Integrate the equation.

1. $x = y'\sin y' + \cos y'$
2. $y' = arctg\,\frac{y}{y'^2}$
3. $x = y' + \ln y'$
4. $\arcsin\frac{x}{y'} = y'$
5. $y = e^{y'}(y' - 1)$
6. $x = 2(\ln y' - y')$

Answers.

1) $\begin{cases} x = p\sin p + \cos p \\ y = (p^2 - 2)\sin p + 2p\cos p + C \end{cases}$

2) $\begin{cases} y = p^2 tgp \\ x = ptgp - lncosp + C \end{cases}$

3) $x = \sqrt{2(y - C) - 1} + \ln [\sqrt{2(y - C) - 1}]$

4) $x = psinp; \ \ y = (p^2 - 1)sinp + pcosp + C$

5) $x = e^p + C, \ \ y = e^p(p - 1) \qquad or \ \ y = (x - C)[\ln(x - C) - 1]$

6) $x = 2(lnp - p); \ \ y = 2p - p^2 + C$

20 - §. Equation that does not contain an explicit independent variable

An equation that does not contain an explicitly independent variable has the form:

$$F(y, y') = 0$$

If this equation can be solved for y:

$$y = f\,(y') \qquad (1)$$

then its solutions can be found using quadratures in parametric form. [9].

Indeed, let $y = \varphi(x)$ − be a solution to equation (1), along which $y' \neq const$. We choose again y' the parameter: $y' = t^2$. Then for this solution

$$y = f(t), \qquad dy = f'(t)dt \ , \qquad dx = \frac{dy}{y} = \frac{f'(t)}{t}dt$$

which means

$$x = \int \frac{f'(t)}{t}dt + C$$

where C- is a number.

Any solution of equation (1) of the indicated type can be written in parametric form:

$$\begin{cases} x = \int \frac{f'(t)}{t} dt + C \\ y = f(t) \end{cases} \qquad (2)$$

By direct substitution into equation (1) we are convinced that for any value of the constant C, relations (2) determine the solution to this equation.
Relations (2), where C is an arbitrary constant, cover all solutions of equation (1), except, perhaps, solutions along which the derivative y′ is constant.
If along some solution $y = \varphi(x)$ the derivative is constant, then this solution has the form

$$y = ax + b$$

where a and b - are some numbers.

Substituting equation (1), we get

$$ax + b \equiv f(a), \; a = 0, \qquad b = f(0).$$

To relation (2), in the case where $f(0)$ makes sense, it is necessary to add the solution $y = f(0)$.

Example. $y = y' + lny'$- an incomplete equation that is unresolved with respect to the derivative. We will solve it in parametric form.

Solution. Let $y' = t$ - parameter. Then

$$y = t + lnt, \qquad dy = (1 + \frac{1}{t})\, dt$$

$$dx = \frac{dy}{t}, \qquad that\ is\ dx = (\frac{1}{t} + \frac{1}{t^2})\, dt$$

which means

$$x = lnt - \frac{1}{t} + C.$$

Ratios

$$\begin{cases} x = lnt - \frac{1}{t} + C \\ y = t + lnt \end{cases}$$

where C is an arbitrary constant, cover all solutions of this equation.

Try to decide for yourself [3]

Solve equations.

1. $y\sqrt{(1 + y'^2)} = y'$
2. $x = e^{2y'}(2y'^2 - 2y' + 1)$
3. $y = y' lny$
4. $x = y'(1 + e^{y'})$
5. $x = 2y' + 3y'^2$

Answers.

1) $x = \ln\left[\frac{\sqrt{1+p^2}-1}{p}\right] + \frac{p}{\sqrt{1+p^2}} + C, \ y = \frac{p}{\sqrt{1+p^2}}$

2) $x = 0{,}5ln^2p + lnp + C, \ y = plnp$

3) $x = p(1 + e^p), \ y = 0{,}5p^2 + (p^2 - p + 1)e^p + C$

4) $x = e^{2p}(2p^2 - 2p + 1), \quad y = e^{2p}(2p^3 - 3p^2 + 3p - 1{,}5) + C$

5) $x = 2p + 3p^2, \quad y = 2p^3 + p^2 + C$

21 - §. Linear homogeneous equations of the second order with constant coefficients

We have a linear homogeneous equation of the second order

$$y'' + py' + qy = 0 \qquad (1)$$

where p and q are constant real numbers. To find the general integral of this equation, it is enough, as was proven above, to find two linearly independent partial solutions. [1]

We will look for private solutions in the form

$$y = e^{kx}, \text{ where } \quad k = const; \ (2)$$

Then

$$y' = ke^{kx}, \ y'' = k^2e^{kx}$$

Substituting the resulting derivative expressions into equation (1), we find

$$e^{kx}(k^2 + pk + q) = 0$$

Because $e^{kx} \neq 0$, that means

$$k^2 + pk + q = 0 \qquad (3)$$

Therefore, if k satisfies equation (3), then e^{kx} will be a solution to equation (1). Equation (3) is called a characteristic equation in relation to equation (1).

The characteristic equation is a quadratic equation with two roots; denote them by k_1 and k_2. Wherein

$$k_1 = -\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\,,\quad k_2 = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}$$

I. k_1 and $k_2$are real and, moreover, unequal numbers ($k_1 \neq k_2$);
II. k_1 and k_2 are complex numbers;
III. k_1 and k_2 are real equal numbers ($k_1 = k_2$).

I. The roots of the characteristic equation are real and different:

$k_1 \neq k_2$. In this case, the particular solutions will be the functions

$$y_1 = e^{k_1 x}\,, \quad y_2 = e^{k_2 x}$$

These solutions are linearly independent, since

$$\frac{y_2}{y_1} = \frac{e^{k_2 x}}{e^{k_1 x}} = e^{(k_2 - k_1)x} \neq \text{const}$$

Therefore, the general integral has the form

$$y = C_1 e^{k_1 x} + C_2 e^{k_2 x}$$

Example 1. Given equation

$$y'' + y' - 2y = 0$$

The characteristic equation has the form

$$k^2 + k - 2 = 0$$

We find the roots of the characteristic equation:

$$k_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2}, \qquad k_1 = 1\,,\ k_2 = -2$$

Generalintegralflattery

$$y = C_1 e^{x} + C_2 e^{-2x}. \qquad \blacksquare$$

II. The roots of the characteristic equation are complex.

Since complex roots are pairwise conjugate, we denote

$$k_1 = \alpha + i\beta, \quad k_2 = \alpha - i\beta$$

where $\alpha = -\frac{p}{2}$, $\beta = \sqrt{q - \frac{p^2}{4}}$

Particular solutions can be written in the form

$$y_1 = e^{(\alpha+i\beta)x}, \; y_2 = e^{(\alpha-i\beta)x} \qquad (4)$$

These are complex functions of the real argument that satisfy the differential equation (1).

If any complex function has a real argument

$$y = u(x) + iv(x) \qquad (5)$$

satisfies equation (1), then this equation is satisfied by the functions $u(x)$ and $v(x)$.

Indeed, substituting expression (5) into equation (1), we will have

$$[u(x) + iv(x)]'' + p[u(x) + iv(x)]' + q[u(x) + iv(x)] \equiv 0$$

or

$$(u'' + pu' + qu) + i(v'' + pv' + qv) \equiv 0$$

But the complex function is equal to zero if and only if the real part and the imaginary part are equal to zero, that is

$$u'' + pu' + qu = 0, \qquad v'' + pv' + qv = 0$$

$u(x)$ and $v(x)$ are solutions of the equation.

Let us rewrite complex solutions (4) in the form of the sum of the real imaginary part:

$$y_1 = e^{\alpha x}cos\beta x + ie^{\alpha x}sin\beta x \; , \; y_2 = e^{\alpha x}cos\beta x - ie^{\alpha x}sin\beta x$$

As shown by partial solutions of equation (1), there will be real functions

$$\tilde{y}_1 = e^{\alpha x}cos\beta x, \qquad \tilde{y}_2 = e^{\alpha x}sin\beta x \qquad (6)$$

The functions are linearly independent, since $\tilde{y}_1$ and $\tilde{y}_2$

$$\frac{\bar{y}_1}{\bar{y}_2} = \frac{e^{\alpha x}\cos\beta x}{e^{\alpha x}\sin\beta x} = ctg\beta x \neq const$$

Consequently, the general solution to equation (1) in the case of complex roots of the characteristic equation has the form

$$y = C_1\bar{y}_1 + C_2\bar{y}_2 = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x$$

or

$$y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x) \qquad (7)$$

where C_1 and C_2 are arbitrary constants.

An important special case of solution (7) is the case when the roots of the characteristic equation are pure imaginary.

This occurs when in equation (1) $p = 0$, the ion has the form

$$y'' + qy = 0$$

The characteristic equation (3) takes the form

$$k^2 + q = 0, \quad q > 0$$

Roots of the characteristic equation

$$k_{1,2} = \pm i\sqrt{q} = \pm i\beta, \qquad \alpha = 0$$

Solution (7) takes the form

$$y = C_1\cos\beta x + C_2\sin\beta x$$

Example 2. Given equation

$$y'' + 2y' + 5y = 0$$

Find a general integral and a particular solution that satisfies the initial conditions $y|_{x=0} = 0$, $y'|_{x=0} = 1$. Construct a graph.

Solution.1) write the characteristic equation

$$k^2 + 2k + 5 = 0$$

and let's find its roots

$$k_1 = -1 + 2i, \qquad k_2 = -1 - 2i$$

hence the general integrality

$$y = e^{-x}(C_1 cos2x + C_2 sin2x)$$

2) we will find a particular solution that satisfies these initial conditions and determine the corresponding values of C_1 and C_2.
Based on the first condition we find:

$$0 = (e^{-0} C_1 \cos (2 \cdot 0) + C_2 \sin (2 \cdot 0))$$

whence $C_1 = 0$. Noticing that

$$y' = e^{-x} 2C_2 cos2x - e^{-x} C_2 sin2x)$$

from the second condition we obtain $1 = 2C_2$, that is, $C_2 = \frac{1}{2}$. Thus, the required particular solution is $y = \frac{1}{2} e^{-x} sin2x$. ■

***III.* The roots of the characteristic equation are real and equal.**

In this case $k_1 = k_2$. One particular solution $y_1 = e^{k_1 x}$ is obtained based on previous reasoning. We need to find a second particular solution that is linearly independent of the first.
We will look for the second particular solution in the form

$$y_2 = u(x) e^{k_1 x}$$

where u(x) is an unknown function to be determined.

Differentiating, we find

$$y_2' = u' e^{k_1 x} + k_1 u e^{k_1 x} = e^{k_1 x}(u' + k_1 u)$$

$$y_2'' = u'' e^{k_1 x} + 2k_1 u' e^{k_1 x} + k_1^2 u e^{k_1 x} = e^{k_1 x}(u'' + 2k_1 u' + k_1^2 u)$$

Substituting expressions for derivatives into equation (1), we obtain

$$e^{k_1 x}[u'' + (2k_1 + p)u' + (k_1^2 + pk_1 + q)u] = 0$$

Since k1 is a multiple root of the characteristic equation, then

$$k_1^2 + pk_1 + q = 0$$

In addition, $k_1 = k_2 = -\frac{p}{2}$ or $2k_1 = -p$, $2k_2 + p = 0$,

There fore, in order to find u(x), we need to solve the equation $u''e^{k_1 x} = 0$ or $u'' = 0$. Integrating, we get $u = Ax + B$.

In particular, we can put $A = 1$, $B = 0$; then $u = x$. Thus, as a second particular solution we can take

$$y_2 = xe^{k_1 x}$$

This solution is linearly independent of the first one, since

$$\frac{y_2}{y_1} = x \neq const$$

Therefore, the general integral will be the function

$$y = C_1e^{k_1 x} + C_2xe^{k_1 x} = e^{k_1 x}(C_1 + C_2x) \,. \qquad \blacksquare$$

Example 3. Given equation

$$y'' - 4y' + 4y = 0$$

We write the characteristic equation $k^2 - 4k + 4 = 0$. Find its roots: $k_1 = k_2 = 2$. We will use the general integral

$$y = C_1e^{2x} + C_2xe^{2x}. \qquad \blacksquare$$

Try to decide for yourself [3]

1. Show that $y = C_1e^{3x} + C_2e^{-3x}$ is a general solution to the equation
$$y'' - 9y = 0.$$
2. Given the equation $y''' - y' = 0$. Do the functions e^x, e^{-x}, chx, which are, as can be easily verified, solutions to this equation, constitute a fundamental system of solutions?
3. Theequation $y'' - y = 0$ is satisfied by two partial solutions $y_1 = shx$, $y_2 = chx$. Do they constitute a fundamental system?

Answers.

1) $y = C_1e^{3x} + C_2e^{-3x} - common\ decision$

2) $chx = \frac{e^x+e^{-x}}{2}$, these three functions are linearly dependent.

3) Yes

II- Chapter. Differential equations of higher order

1 - §.Linear homogeneous equations of the nth order with constant coefficients (Vandermonde method)

Consider a linear homogeneous equation of the nth order

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = 0 \qquad (1)$$

or in short, $L(y) = 0$. [9].
Where $L(y) = y^{(n)}+a_1y^{(n-1)} + \cdots + a_ny$ - is a linear homogeneous equation with real coefficients, a $a_1,\ a_2,\ \ldots\ a_n$ −an are constants.We will look for solutions to this equation in the form $y = e^{kx},\ where\ k$ - a certain number.

Because

$$y' = ke^{kx},\ y'' = k^2e^{kx}, \ldots, y^{(n)} = k^{(n)}e^{kx}, \text{ that}$$

$$L(e^{kx}) = e^{kx}[k^n + a_1k^{n-1} + \cdots + a_n]$$

Polynomial $F(k) = k^n + a_1k^{n-1} + \cdots + a_n$

is called the characteristic polynomial of the differential equation (1).

In order for a function to be a solution to equation (1), it is necessary and sufficient that $y = e^{kx}$

$$L(e^{kx}) = 0, \ \ that \ is \quad e^{kx}F(k) = 0.$$

The multiplier is nonzero, the number k must satisfy the equation e^{kx}

$$F(k) = 0 \quad (2)$$

The equation $F(k) = 0$ is called the characteristic equation corresponding to this differential equation (1). The function $y = e^{kx}$ was a solution to the differential equation (1), it is necessary and sufficient that the number$k$be the root of the corresponding characteristic equation (2).

Characteristic equation (2)

$$k^n + a_1k^{n-1} + \cdots + a_n = 0$$

is an algebraic equation of nth degree with respect to k, according to the basic theorem of algebra, it has n roots: $k_1, k_2, \ldots, k_n$. Each of these roots corresponds to a solution to differential equation (1).

1. ***All roots of characteristic equation (2) are real and different.***

The functions $y_1 = e^{k_1x}, y_2 = e^{k_2x}, \ldots, y_n = e^{k_nx}$ are the proven solutions to the differential equation (1).To prove this, let's compile the Wronski determinant:

$$W(x) = \begin{vmatrix} e^{k_1x} & e^{k_2x} \ldots & e^{k_nx} \\ k_1e^{k_1x} & k_2e^{k_2x} \ldots. & k_ne^{k_nx} \\ \ldots\ldots & \ldots\ldots.. & \ldots\ldots. \\ k_1^{n-1}e^{k_1x} & k_2^{n-1}e^{k_2x} \ldots. & k_n^{n-1}e^{k_nx} \end{vmatrix} = e^{(k_1+k_2+\cdots+k_n)} \cdot V$$

where

$$V = \begin{vmatrix} 1 & 1 \ldots & 1 \\ k_1 & k_2 \ldots & k_n \\ \ldots & \ldots. & \ldots. \\ k_1^{n-1} & k_2^{n-1} \ldots. & k_n^{n-1} \end{vmatrix}$$

- determinant known as the Vandermonde determinant.

It is equal to the product of all possible differences of the first powers of its elements.

$V = (k_2 - k_1)(k_3 - k_1) \cdots (k_n - k_1)(k_3 - k_2) \cdots (k_n - k_2) \cdots (k_n - k_{n-1})$.

Since the roots $k_1, k_2, \dots, k_n$ are different, the Vandermonde determinant, and therefore the Wronski determinant, are different from zero.In the case of different real roots of the characteristic equation (2) solutions $k_1, k_2, \dots, k_n$

$$y_1 = e^{k_1 x},\ y_2 = e^{k_2 x}, \dots, y_n = e^{k_n x}$$

constitute a fundamental system, which means that the general solution to equation (2) can be written as:

$$y = C_1 e^{k_1 x} + C_2 e^{k_2 x} + \cdots + C_n e^{k_n x}$$

where $C_1, C_2, \dots, C_n$ - are arbitrary constants.

Example 1. Given the equation $y'' - 3y' + 2y = 0$. The characteristic equation $k^2 - 3k + 2 = 0$ has the form, its roots $k_1 = 1,\ k_2 = 2$ are real and different, corresponding to particular solutions of the equation $y_1 = e^x,\ y_2 = e^{2x}$.

Common decision:

$$y = C_1 e^x + C_2 e^{2x}. \ \blacksquare$$

where C_1, C_2 – are arbitrary constants.

Example 2. Given an equation $y''' - 7y'' + 6y' = 0$, we compose a characteristic equation:

$$k^3 - 7k^2 + 6k = 0,\ \ k(k^2 - 7k + 6) = 0,\ \ k_1 = 0,\ \ k_2 = 1,\ \ k_3 = 6.$$

The roots are real and different. The corresponding solutions are:

$$y_1 = 1,\ y_2 = e^{2x},\ \ y_3 = e^{6x}$$

General solution of the equation:

$$y = C_1 + C_2 e^x + C_3 e^{6x}. \ \blacksquare$$

where $C_1 C_2, C_3$ –are arbitrary constants.

2. ***All roots of the characteristic equation are real, but among them there are multiple roots.***

In this case, among the $k_1, k_2, \dots, k_n$ numbers, there will be less than n different roots, and accordingly $e^{k_1 x}, e^{k_2 x}, \dots, e^{k_n x}$, among the solutions, there will also be less than n different solutions. This means that the system of these decisions will no longer be fundamental.

To obtain the missing solutions, you can use the operator properties:

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$$

with constant coefficients.

Equality $L(e^{kx}) = e^{kx}F(k)$ holds for all values of the number k. Let's differentiate$m$ it differently$k$:

$$\frac{\partial^m L(e^{kx})}{\partial k^m} = \frac{\partial^m [e^{kx}F(k)]}{\partial k^m}$$

Wherein

$$\frac{\partial^m L(e^{kx})}{\partial k^m} = \frac{\partial}{\partial k^m} [(e^{kx})^n + a_1(e^{kx})^{(n-1)} + \cdots + a_n e^{kx}]$$

Taking into account the properties of derivatives, as well as the theorem of independence of the mixed derivative from the sequence of differentiations, we obtain:

$$\frac{\partial^m L(e^{kx})}{\partial k^m} = \left(\frac{\partial^m e^{kx}}{\partial k^m}\right)^{(n)} + a_1 \left(\frac{\partial^m e^{kx}}{\partial k^m}\right)^{(n-1)} + \cdots + a_n \frac{\partial^m e^{kx}}{\partial k^m} = L\left(\frac{\partial^m e^{kx}}{\partial k^m}\right) =$$

$$= L(x^m e^{kx})$$

On the other hand, using Leibniz's rule to calculate the derivative of order m of the product of two functions, we find:

$$\frac{\partial^m [e^{kx}F(k)]}{\partial k^m} = \frac{\partial^m e^{kx}}{\partial k^m} F(k) + m \frac{\partial^{m-1} e^{kx}}{\partial k^{m-1}} F'(k) + C_m^2 \frac{\partial^{m-2} e^{kx}}{\partial k^{m-2}} F'' + \cdots + e^{kx} F^{(m)}(k) =$$

$$= e^{kx}[x^m F(k) + m x^{m-1} F'(k) + C_m^2 x^{m-2} F''(k) + \cdots + F^{(m)}(k)]$$

Equating the results obtained, we arrive at the formula

$$L(x^m e^{kx}) = e^{kx}[x^m F(k) + m x^{m-1} F'(k) + C_m^2 x^{m-2} F''(k) + \cdots + F^{(m)}(k)] \quad (3)$$

valid for any natural number m and any number k.

Let the k_1 – root of equation (2) be multiplicity ym_1. This means that the characteristic polynomial F(k) can be written in the form:

$$F(k) = (k - k_1)^{m_1}\Phi(k), \quad \text{where } \Phi(k_1) \neq 0$$

Since every multiple root of a polynomial is the root of a multiplicity of one lesser for its derivative, we conclude that

$$F(k_1) = F'(k_1) = \cdots = F^{(m_1-1)}(k_1) = 0, \ a\, F^{(m_1)}(k_1) \neq 0$$

Substituting into formula (3) instead of k the number k_1, instead of m successively the numbers 0, 1, 2, ...,$m_1 - 1$, we obtain that $L(x^m e^{k_1 x}) = 0$. This means that the functions

$$e^{k_1 x}, xe^{k_1 x}, x^2 e^{k_1 x}, \dots, x^{m_1-1} e^{k_1 x}$$

are solutions to this differential equation. The root k_1 of the multiplicity $m_1$of the characteristic equation (2) is put into m_1 correspondence with exactly different solutions of the differential equation.

Example 3. Given the equation $y'' - 4y' + 4y = 0$. Its characteristic equation

$$k^2 - 4k + 4 = 0$$

has multiple roots $k_1 = k_2 = 2$. The corresponding partial solutions of the equation have the form:

$$y_1 = e^{2x}, \qquad y_2 = xe^{2x}$$

Common decision:

$$y = C_1 e^{2x} + C_2 x e^{2x}.$$

where C_1 and C_2 – are arbitrary constants.
Let's find a particular solution to this equation using the initial data $0, -1, 1$.
Because

$$y = C_1 e^{2x} + C_2 x e^{2x}, \qquad where\ y' = 2C_1 e^{2x} + C_2 e^{2x}(1 + 2x).$$

at $x = 0$ we have

$$-1 = C_1, 1 = 2C_1 + C_2, \text{ from where } C_1 = -1, C_2 = 3$$

The required particular solution has the form:

$$y = -e^{2x} + 3xe^{2x}.$$

Example 4. An equation $y^V - 9y''' = 0$ is given. The characteristic equation has the form:

$$k^5 - 9k^3 = 0$$

his roots

$$k^3(k^2 - 9) = 0, \quad k_1 = k_2 = k_3 = 0, \; k^2 = 9\,, \; k_{4,5} = \pm 3.$$

The root $k_1 = k_2 = k_3 = 0$ – is threefold, the roots $k_4 = 3, k_5 = -3$ are simple.

3. ***Among the roots of the characteristic equation (2) there are imaginary roots.***

If each value of the real variable x is assigned a complex number, the real numbers $y = u + iv,$ that u and v are given by the complex function of the real arguments x:

$$y = f(x) \;\; or \;\; y = u(x) + iv(x)$$

In this case, the functions $u\ (x)$ and $v\ (x)$ are called, respectively, the real part of the function $y = f\ (x)$.

For complex functions of a real variable, one can introduce the concept of limit, continuity, and derivative in a similar way to how it was done in the actual case. From the definition of a derivative, in particular, it follows that

$$y' = u'(x) + iv'(x), \dots, y^{(n)} = u^{(n)}(x) + iv^{(n)}(x)$$

Consider linear homogeneous equation (1) with real coefficients:

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y$$

A complex function of a real variable is called a solution to this equation if $y = u(x) + iv(x)$

$$L(y) \equiv 0, \qquad that\ is\ L(u(x) + iv(x)) \equiv 0$$

Substituting $y = u(x) + iv(x)$ equation (1), we get:

$$(u + iv)^{(n)} + a_1(u + iv)^{(n-1)} + \dots + a_n(u + iv) =$$

$$= [u^{(n)} + a_1 u^{(n-1)} + \dots + a_n u] + i[v^{(n)} + a_1 v^{(n-1)} + \dots + a_n v] =$$

$$= L(u) + iL(v) \equiv 0$$

Since a complex number is equal to zero only when its real and imaginary parts are equal to zero, we conclude that

$$L(u(x)) \equiv 0 \ \ and \ \ L(v(x)) \equiv 0.$$

If a function $y = u(x) + iv(x)$ is a solution to equation (1), then its real imaginary parts are also solutions to equation (1). For any real numbers α and β, we define the complex exponential function of the real argument by the equality:

$$e^{(\alpha+i\beta)x} = e^{\alpha x}(cos\beta x + isin\beta x)$$

If it $\alpha = 0$, follows from this that for any real number β the equality holds,

$$e^{i\beta x} = cos\beta x + isin\beta x$$

Substituting the number $-\beta$ instead of the number β, we get the equality

$$e^{-i\beta x} = cos\beta x - isin\beta x$$

From these equalities we obtain as a consequence that

$$cos\beta x = \frac{e^{i\beta x}+e^{-i\beta x}}{2}, \ sin\beta x = \frac{e^{i\beta x}-e^{-i\beta x}}{2}.$$

A function $e^{(\alpha+i\beta)x}$ for any value has derivatives of all orders, and:

$$\left(e^{(\alpha+i\beta)x}\right)' = (\alpha + i\beta)e^{(\alpha+i\beta)x},$$

$$\left(e^{(\alpha+i\beta)x}\right)'' = (\alpha + i\beta)^2 e^{(\alpha+i\beta)x},$$

....

$$\left(e^{(\alpha+i\beta)x}\right)^{(n)} = (\alpha + i\beta)^n e^{(\alpha+i\beta)x}.$$

Let be $k_1 = \alpha_1 + i\beta_1$ - a simple root of the characteristic equation (2). This equation has real coefficients, and among its roots there is a root that is the complex conjugate of root k_1. Denoted by k_2:

$$k_2 = \alpha_1 - i\beta_1$$

Roots k_1 and k_2 correspond to complex solutions

$$y_1 = e^{(\alpha_1+i\beta_1)x} \quad and \ \ y_2 = e^{(\alpha_1-i\beta_1)x}$$

differential equation (1).

It has been shown that the real imaginary parts of these solutions, in turn, are solutions to equation (1). Because

$$y_1 = e^{\alpha_1 x}cos\beta_1 x + ie^{\alpha_1 x}sin\beta_1 x, \qquad y_2 = e^{\alpha_1 x}cos\beta_1 x - ie^{\alpha_1 x}sin\beta_1 x$$

then this means that the functions

$$e^{\alpha_1 x}cos\beta_1 x, \qquad e^{\alpha_1 x}sin\beta_1 x \quad and \ - e^{\alpha_1 x}sin\beta_1 x$$

will be real solutions to equation (1). Discarding the last of them, we obtain two valid solutions to equation (1):

$$\tilde{y} = e^{\alpha_1 x}cos\beta_1 x \qquad and \ \tilde{y} = e^{\alpha_1 x}sin\beta_1 x$$

corresponding to two simple complexes with conjugate roots $k_{1,2} = \alpha_1 \pm i\beta_1$ of the characteristic equation. If a number $k_1 = \alpha_1 + i\beta_1$ is a root of the characteristic equation (2) of multiplicity m_1, then the complex conjugate number $k_2 = \alpha_1 - i\beta_1$ is also a root of the equation (2) of multiplicity m_1. These roots $2m_1$ correspond to complex solutions of the differential equation (1):

$$e^{(\alpha_1+i\beta_1)x}, \quad xe^{(\alpha_1+i\beta_1)x}, \dots, x^{m_1-1}e^{(\alpha_1+i\beta_1)x},$$

$$e^{(\alpha_1-i\beta_1)x}, \qquad xe^{(\alpha_1-i\beta_1)x}, \dots, x^{m_1-1}e^{(\alpha_1-i\beta_1)x}$$

Based on complex solutions, separating their real imaginary parts, we can create a system of $2m_1$ real solutions to the same equation (1):

$$e^{\alpha_1 x}cos\beta_1 x, \qquad xe^{\alpha_1 x}cos\beta_1 x, \dots, \ x^{m_1-1}e^{\alpha_1 x}cos\beta_1 x$$

$$e^{\alpha_1 x}sin\beta_1 x, \qquad xe^{\alpha_1 x}sin\beta_1 x, \dots, \ x^{m_1-1}e^{\alpha_1 x}sin\beta_1 x$$

General rule for solving a linear homogeneous differential equation with constant coefficients.

1. ***We compose a characteristic equation and find all its roots.***
2. ***We find particular solutions to this differential equation, and:***

a) each simple real root k of the characteristic equation is associated with a solution e^{kx},

b) each *m* - multiple real rootk of the characteristic equation is put in correspondence with *m* solutions:

$$e^{kx}, xe^{kx}, x^2e^{kx}, \dots, x^{m-1}e^{kx}$$

c) each pair of simple complex conjugate roots $\alpha \pm i\beta$ of the characteristic equation is associated with two solutions:

$$e^{\alpha x}\cos\beta x \quad and \; e^{\alpha x}\sin\beta x$$

d) each pair of m – multiple complex conjugate roots of the characteristic equation is put in correspondence with $2m$ solutions:

$$e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, \dots, x^{m-1}e^{\alpha x}\cos\beta x$$

$$e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, \dots, x^{m-1}e^{\alpha x}\sin\beta x$$

The set of solutions obtained in this way forms a fundamental system of solutions to the equation in the section $-\infty < x < +\infty$.

3. ***We compose a linear combination of the solutions found***.

This linear combination of solutions with arbitrary coefficients will give a general solution to the equation in the XOY plane.

Example 5. Given the equation $y''' - y = 0$. Characteristic equation

$k^3 - 1 = 0$ has roots $k_1 = 1,\; k_{2,3} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

These roots correspond to the solutions

$$y_1 = e^x, \; y_2 = e^{-\frac{x}{2}}\cos\frac{\sqrt{3}}{2}x, \qquad y_3 = e^{-\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x$$

General solution of the equation:

$$y = C_1e^x + e^{-\frac{x}{2}}\left[C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right]$$

where C_1, C_2, C_3 – are arbitrary constants.

Example 6. If the equation $L(y) = 0$ with constant coefficients has roots of the characteristic equation of the number

$$k_{1,2,3,4} = -1, \qquad k_{5,6} = 2 + 3i, \; k_{7,8} = 2 - 3i,$$

then the general solution of this equation has the form:

$$y = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x} + C_4x^3e^{-x} + C_5e^{2x}\cos 3x +$$

$$+C_6xe^{2x}\cos 3x + C_7e^{2x}\sin 3x + C_8xe^{2x}\sin 3x$$

where $C_1, C_2, \dots, C_8$ −are arbitrary constants.

(Find using the usual method)

Definition1. If for all$x$ of the interval $[a, b]$ the equality

$$\varphi_n(x) = A_1\varphi_1(x) + A_2\varphi_2(x) + \cdots + A_{n-1}\varphi_{n-1}(x)$$

where $A_1, A_2, \dots, A_{n-1}$ −are constant numbers that are not all equal to zero, so that $\varphi_n(x)$ is expressed linearly through the functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_{n-1}(x)$.

Definition 2. n functions $\varphi_1(x)$, $\varphi_2(x)$, ... , $\varphi_{n-1}(x)$, $\varphi_n(x)$ are called linearly independent if none of these functions can be expressed linearly through the others.

Note1. From the definitions it follows that if the functions $\varphi_1(x)$, $\varphi_2(x)$, ... , $\varphi_n(x)$ are linearly dependent, then there are constants C_1, C_2, ... , C_n, not all equal to zero, such that for all the segment $[a, b]$ the identity will hold

$$C_1\varphi_1(x) + C_2\varphi_2(x) + \cdots + C_n\varphi_n(x) \equiv 0$$

Example 1. The functions $y_1 = e^x$, $y_2 = e^{2x}, y_3 = e^{3x}$ are linearly independent, since neither for $C_1 = 1, C_2 = 0$, $C_3 = -\frac{1}{3}$ the identity holds

$$C_1e^x + C_2e^{2x} + C_3e^{3x} \equiv 0$$

Example 2. The functions $y_1 = 1$, $y_2 = x, y_3 = x^2$ are linearly independent, since no C_1, C_2, C_3, are simultaneously equal to zero, the expression

$$C_1 \cdot 1 + C_2x + C_3x^2$$

will not be identically zero.

Example 3. Functions $y_1 = e^{k_1x}, y_2 = e^{k_2x}$, …, $y_n = e^{k_nx}$,…
where $k_1, k_2, \dots, k_n$ − are distinct numbers, linearly independent.

Let us now move on to solving equation (1). The following theorem holds for this equation.

Theorem. If the functions $y_1, y_2, \dots, y_n$ are linearly independent solutions of equation (1), then its general solution is

$$y = C_1y_1 + C_2y_2 + \cdots + C_ny_n \qquad (2)$$

where $C_1, \dots, C_n$ – are arbitrary constants.

If the coefficients of equation (1) are constant, then the general solution is found in the same way as in the case of a second-order equation.

1) Making up a characteristic equation

$$k^n + a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_n = 0.$$

2) Find the roots of the characteristic equation $k_1, k_2, \dots, k_n$.

3) Based on the nature of the roots, we write out particular linearly independent solutions, guided by the fact that:

a) each real single root k corresponds to a particular solution e^{kx};

b) each pare complex conjugate single root

$k^{(1)} = \alpha + i\beta$ and $k^{(2)} = \alpha - i\beta$ correspond to two partial solutions $e^{\alpha x} cos\beta x \ and \ e^{\alpha x} sin\beta x$;

c) each real root k of multiplicity corresponds to linearly independent partial solutions $e^{kx}, xe^{kx}, \dots, x^{r-1}e^{kx}$;

d) each pair of complex conjugate roots

$$k^{(1)} = \alpha + i\beta, \quad k^{(2)} = \alpha - i\beta$$

multiplicities μ correspond to 2μ particular solutions

$$e^{\alpha x} cos\beta x, \ xe^{\alpha x} cos\beta x, \dots, x^{\mu-1} e^{\alpha x} cos\beta x$$

$$e^{\alpha x} sin\beta x, xe^{\alpha x} sin\beta x, \dots, \ x^{\mu-1} e^{\alpha x} sin\beta x$$

4) Having found n linearly independent partial solutions to $y_1, y_2, \dots, y_n$ we construct a general solution to this linear equation

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

where $C_1, \dots, C_n$ - are arbitrary constants.

Example 4. Find a general solution to the equation

$$y^{IV} - y = 0$$

Solution. Making up a characteristic equation

$$k^4 - 1 = 0$$

we find the roots of the characteristic equation:

$$k_1 = 1, \qquad k_2 = -1, \qquad k_3 = i, \qquad k_4 = -i$$

write a general integral

$$y = C_1e^x + C_2e^{-x} + C_3cosx + C_4sinx$$

Where C_1, C_2, C_3, C_4 – are arbitrary constants.

Try to decide for yourself [3]

1. Find the general solution to the equation $y'' - 7y' + 6y = 0$
2. Find the general solution to equation $y^{IV} - 13y'' + 36y = 0$
3. Find a general solution to the equation $y''' - 2y'' + y' = 0$
4. Find a general solution to the equation $y'' - 4y' + 13y = 0$

Answers.1) $y = C_1e^{6x} + C_2e^x$

2) $y = C_1e^{3x} + C_2e^{-3x} + C_3e^{2x} + C_4e^{-2x}$

3) $y = C_1 + C_2e^x + C_3xe^x$

4) $y = e^{2x}(C_1cos3x + C_2sin3x)$

2 - §. Higher order differential equations (general concepts)

Differential equation of nth order called ratio

$$F(x, y, y', y'', \dots, y^{(n)}) = 0 \qquad (1)$$

Connecting the independent variable, the desired function and its derivatives up to the n -th order inclusive. Any differential equation of order higher than first is called a higher order equation. [9].

For example, equations

$$y''' - 1 = 0, \qquad xy' - 2y'' + y''^3 = 0, \qquad y^{V^3} - e^{4y'''} + 1 = 0$$

– equations of higher (third, second and fifth, respectively) orders.

In some cases, equation (1) can be resolved with respect $y^{(n)}$,to i.e. in the form:

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}) = 0 \quad (2)$$

Such an equation is called an nth-order equation resolved with respect to the highest derivative.

Second order differential equation

$$F(x, y, y', y'') = 0 \qquad (3)$$

expresses the relationship between the coordinates of the point of the integral curve, the angular coefficient of its tangent and the curvature at this point. Integral curves of equation (3) are curves that at each point have the relationship prescribed by the equation between the angular coefficient of the tangent to the curve and the curvature.

Differential equations are widely used in mechanics, regardless of the specific physical or geometric meaning of the argument x of the desired function y, numbers x_0, y_0, y'_0 , representing a certain value of the argument $(x = x_0)$ and the value of the desired function $(y = y_0)$ and its derivative $(y' = y'_0)$ in this case, the values of the argument are usually called initial conditions or initial data for the equation and second order:

$$F(x, y, y', y'') = 0$$

The solution $y = \varphi(x)$ of this equation satisfies the initial conditions x_0, y_0, y'_0, if $\varphi(x_0) = y_0$, $\varphi'(x_0) = y'_0$.

Geometrically, this integral curve of the equation passes through the point (x_0, y_0) of the XOY plane and has a tangent at this point with an angle coefficient y'_0.

Higher order equations that can be solved with respect to the higher derivative. For these equations there is a theorem on the existence and uniqueness of a solution, similar to the corresponding theorem on the solution of a first-order equation.

Cauchy's theorem. If a function of $(n + 1)$ – variables $f(x, y, y', y'', \dots, y^{(n-1)})$outside some region of $D(n + 1)$ −dimensional space is continuous and has continuous partial derivatives with respect to y, y', y'', …, $y^{(n-1)}$, then whatever the point is $(x_0, y_0, y'_0, \dots, y_0^{(n-1)})$ of this region, there is a unique solution to the equation $y = \varphi(x)$

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

defined within a certain interval containing point x_0, satisfying the initial conditions $x_0, y_0, y'_0, \dots, y_0^{(n-1)}$.

For an n-th-order equation (1) and (2), the initial conditions are a system of $(n + 1)$ numbers $x_0, y_0, y'_0, \dots, y_0^{(n-1)}$, representing the initial value of the independent variable x $(x = x_0)$ and the values of the sought function y and all derivatives up to $(n - 1)$ th orderinclusive at $x = x_0$.

$$y|_{x=x_0} = y_0, \ y'|_{x=x_0} = y'_0, \dots, y^{(n-1)}|_{x=x_0} = y_0^{(n-1)} \qquad (2)$$

These conditions are called initial conditions.

If we consider the second-order equation

$$y'' = f(x, y, y') \quad (3)$$

where x_0, y_0, y'_0 are given numbers. The geometric meaning of these conditions is as follows: a single curve passes through a given point of the plane (x_0, y_0) with a given tangent of the tangent angle y'_0. From this it follows that if y'_0 we set different values for constant x_0 and y_0, then we will obtain an infinite number of integral curves with different slope angles passing through a given point.

Let us now introduce the concept of a general solution to an n th order equation.

Definition.General solution n th order differential equation is called a function

$$y = \varphi(x, C_1, C_2, \dots, C_n) \quad (4)$$

depending on n arbitrary constants $C_1, C_2, \dots, C_n$ such that:

a) it satisfies the equation for any values of the constants

$$C_1, C_2, \dots, C_n;$$

b) under given initial conditions

$$y|_{x=x_0} = y_0, \; y'|_{x=x_0} = y'_0, \dots, \qquad y^{(n-1)}|_{x=x_0} = y_0^{(n-1)}$$

constants $C_1, C_2, \dots, C_n$ can be selected so that the function $y = \varphi(x, C_1, C_2, \dots, C_n)$ will satisfy these conditions (assuming that the initial values $x_0, y_0, y'_0, \dots, y_0^{(n-1)}$ belong to the region where the conditions for the existence of the solution are met).

The equation

$$\Phi(x, y, C_2, \dots, C_n) = 0, \qquad (5)$$

the implicitly defining general solution is called the general integral of the differential equation.

Any function resulting from the general solution for specific values of the constants C_1, C_2, ... , C_n is called a particular solution. The graph of a particular solution is called the integral curve of a given differential equation.

Each system of values of these parameters corresponds to the equation $\Phi(x, y, C_{10}, C_{20}, \dots, C_{n0}) = 0$, connecting the variables x and y. This equation defines a certain curve on the XOY plane. The set of all such curves is called a family of curves depending on n parameters, given by equation (5), and equation (5) itself is called the equation of this families of curves.

For example, the family of all non-vertical straight lines of the XOU plane has an equation $y = C_1x + C_2$, C_1 *and* C_2 - parameters.

The family of all circles in the XOY plane has the equation

$$(x - C_1)^2 + (y - C_2)^2 = C^2{}_3$$

This family depends on three parameters C_1, C_2 *and* C_3 *etc.*

Solving a differential equation of the nth order means:

1) find its general solution or

2) find that particular solution of the equation that satisfies the given initial conditions.

Example 1. Find a partial solution of the equation $y'' = xe^{-x}$, satisfying the initial conditions $y(0) = 1,\ y'(0) = 0.$

Solution. Let's find a general solution by sequentially integrating this equation:

$$y' = \int xe^{-x}dx = -xe^{-x} - e^{-x} + C_1$$

$$y = \int[-xe^{-x} - e^{-x} + C_1]dx = xe^{-x} + 2e^{-x} + C_1x + C_2$$

or

$$y = (x + 2) + x + e^{-x}C_1C_2$$

Let's use the initial conditions: $1 = 2 + C_2;\ C_2 = -1;\ 0 = -1 + C_1;\ C_1 = 1.$

Consequently, the required particular solution has the form

$$y = (x + 2) + x - 1e^{-x}$$

The same solution can be found in the following way, using immediately given initial conditions:

$$y' = y'(0) + \int_0^x xe^{-x}dx = [-xe^{-x}\ e^{-x}]_0^x e^{-x} - e^{-x} + 1$$

$$y = y(0) + \int_0^x [-xe^{-x} - e^{-x} + 1]dx = 1 + [(x + 2)e^{-x} + x]^x =_0$$

$$= (x + 2)e^{-x} + x - 1$$

Try to decide for yourself [3]

1. $y^{(V)} = cos^2x$; $y(0) = \frac{1}{32}$; $y'(0) = 0$; $y''(0) = \frac{1}{8}$; $y'''(0) = 0$

2. $y''' = xsinx$; $y(0) = 0$; $y'(0) = 0$; $y''(0) = 2$.

3. $y''' sin^4 x = sin\, 2x$

4. $y'' = 2\, sinx\, cos^2 x - sin^3 x$

5. $y''' = xe^{-x}$; $y(0) = 0$, $y'(0) = 2$; $y''(0) = 2$.

Answers.

1) $y = \frac{1}{48}x^4 + \frac{1}{8}x^2 + \frac{1}{32}cos2x$

2) $y = xcosx - 3sinx + x^2 + 2x$

3) $y = lnsinx + C_1x^2 + C_2x + C_3$

4) $y = -\frac{1}{3}sin^3 x + C_1x + C_2$

5) $y = -(x + 3)e^{-x} + \frac{3}{2}x^2 + 3$

3 - §. Higher order equations allowingdowngrading

One of the main methods used when integrating higher-order differential equations is to reduce the order of the equation, that is, reduce the equation by replacing variables to another equation of lower order. [9]

a) Equations of the form $y^{(n)} = f(x)$, where $f(x)$ - is a function continuous on some interval $a < x < b$ of the OX axis, not only allow a reduction in order, but are also integrated in quadratures. For any solution $y = \varphi(x)$ of the equation

$$y^{(n)} = f(x) \qquad (1)$$

lying in the strip $\{a < x < b, -\infty < y < +\infty\}$, sequentially integrating, we obtain:

$$y^{(n-1)} = \int f(x)dx + C_1$$

$$y^{(n-2)} = \int \left(\int f(x)dx + C_1\right) dx + C_2 = \int dx \int f(x)dx + C_1 x + C_2,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$y = \int dx \int dx \dots \int f(x)dx + C_1 \frac{x^{n-1}}{(n-1)!} + C_2 \frac{x^{n-2}}{(n-2)!} + \dots + C_n \qquad (2)$$

where each integral denotes one of the antiderivatives for the integral function, and $C_1, C_2, \dots, C_n$ −some are constants.

By direct substitution into equation (1), we make sure that function (2) satisfies equation (1) for any values of the $C_1, C_2, \dots, C_n$ constants.

Example. Given the equation $y''' = e^{2x}$. It is required to find a particular solution that satisfies the initial conditions:

$$x_0 = 0,\ y_0 = 1,\ y'_0 = -1,\ y''_0 = 0.$$

Since the function $f(x) = e^{2x}$ is continuous everywhere, the general solution of the equation in the XOY plane is obtained by three times sequential integration of the relation $y''' = e^{2x}$.

$$y'' = \frac{1}{2}e^{2x} + C_1$$

$$y' = \frac{1}{4}e^{2x} + C_1 x + C_2$$

$$y = \frac{1}{8}e^{2x} + C_1\frac{x^2}{2} + C_2 x + C_3.$$

General solution in the XOY plane:

$$y = \frac{1}{8}e^{2x} + C_1\frac{x^2}{2} + C_2 x + C_3,$$

where C_1, C_2, C_3 –are arbitrary constants.

Substituting the initial values int y, y', y'' the expressions, we obtain the following relations to determine the constants C_1, C_2, C_3:

$$1 = \frac{1}{8} + C_3, \quad -1 = \frac{1}{4} + C_2, \quad 0 = \frac{1}{2} + C_1$$

where is it from?

$$C_1 = -\frac{1}{2}, \quad C_2 = -\frac{5}{4}, \quad C_3 = -\frac{7}{8}.$$

The particular solution you are looking for:

$$y = \frac{1}{8}e^{2x} - \frac{1}{4}x^2 - \frac{5}{4}x + \frac{7}{8}. \quad ■$$

General form of a 2nd order differential equation:

$$F(x, y, y', y'') = 0 \quad (1)$$

Let us note 3 particular types of equation (1), when its solution is reduced to the sequential solution of two differential equations of the 1st order.

1. ***The equation does not contain the desired function y,*** that is, it has the form

$$F(x, y', y'') = 0 \quad (2)$$

In this case, we introduce a new unknown function z by putting

$$y' = z$$

Then $y'' = z'$ and (2) takes the form

$$F(x, z, z') = 0$$

first-order equation for z. Having solved it, we find

$$z = \varphi(x, C_1)$$

that is

$$y' = \varphi(x, C_1)$$

then

$$y = \int \varphi(x, C_1)dx + C_2.$$

Example. $y'' - \frac{y'}{x} = xe^x$. Setting $y' = z$, we obtain a linear differential equation of 1st order

$$z' - \frac{z}{x} = xe^x.$$

We'll find

$$z = (e^x + C_1)x$$

From here

$$y = \int(e^x + C_1)xdx = xe^x - e^x + C_1\frac{x^2}{2} + C_2.$$

2. ***The equation does not contain the independent variable x,*** that is, it looks like

$$F(y, y', y'') = 0 \qquad (3)$$

In this case, accept the unknown function $y' = z$ and accept the new independent variable y. Then

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dy} \cdot \frac{dy}{dx} = \frac{dz}{dy}z$$

Equation (3) is transformed into an equation of 1st order

$$F\left(y, z, z\frac{dz}{dy}\right) = 0.$$

Decide it

$$z = \varphi(y, C_1)$$

that is

$$\frac{dy}{dx} = \varphi(y, C_1)$$

where

$$\frac{dy}{\varphi(y, C_1)} = dx$$

and

$$\int \frac{dy}{\varphi(y, C_1)} = x + C_2$$

This is the general integral of a differential equation.

Example. $yy'' - 2y'^2 = 0$. Assuming we get $y' = z,\ y'' = z\frac{dz}{dy}$

$$yz\frac{dy}{dx} - 2z^2 = 0$$

or

$$z\left(y\frac{dz}{dy} - 2z\right) = 0$$

This differential equation splits into two:

$$z = 0, \qquad y\frac{dz}{dy} - 2z = 0$$

The first of $y' = 0$ them $y = C$ gives. In the second, the variables are separated:

$$\frac{dz}{z} = \frac{2dy}{y},$$

where

$$lnz = 2lny + lnC_1$$

$$z = C_1y^2.$$

Remembering what we get $z = \frac{dy}{dx}$

$$\frac{dy}{y^2} = C_1dx$$

And

$$-\frac{1}{y} = C_1x + C_2$$

that is (replacing C_1 and C_2 with $-C_1$ and $-C_2$)

$$y = \frac{1}{C_1x + C_2} \qquad (3)$$

This is the general solution of the differential equation. The previously found solution $y = C$ is contained in (3), obtained from (3) with $C_1 = 0$.

3. The equation has the form

$$y'' = f(y) \qquad (4)$$

This is a special case of equation (2), and therefore it can be solved by replacing

$$y'' = z\frac{dz}{dy}$$

where $z = y'_x$ This substitution transforms the differential equation (4) into the equation

$$z\frac{dz}{dy} = f(y) \qquad (5)$$

giving

$$\frac{z^2}{2} = \int f(y)dy + C_1$$

From here

$$z = \pm\sqrt{2\left[C_1 + \int f(y)dy\right]}$$

or

$$\frac{dy}{\sqrt{2\left[C_1 + \int f(y)dy\right]}} = \pm dx \qquad (6)$$

and another integration leads to the general integral of equation (4).

Example. $y'' = \frac{3}{2}y^2$, $y|_{x=3} = 1$, $y'|_{x=3} = 1$. Making the replacement, we find $y'' = z\frac{dz}{dy}$

$$2zdz = 3y^2dy,$$

where

$$z^2 = y^3 + C_1$$

Assuming here x = 3 and taking into account that in this case it will be $y' = z = 1$, that before the radical we must choose the sign + and that $C_1 = 0$.

$$\frac{dy}{dx} = \sqrt{y^3}.$$

Means,

$$y^{-\frac{3}{2}}dy = dx$$

and

$$-2y^{-\frac{1}{2}} = x + C_2.$$

At $x = 3$ we find $-2 = 3 + C_2$, from which $C_2 = -5$.

$$\frac{2}{\sqrt{y}} = 5 - x$$

and finally

$$y = \frac{4}{(x-5)^2}. \quad \blacksquare$$

b) Equations that do not explicitly contain the sought function and its derivatives up to order k - 1, equations of the form

$$F(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}) = 0$$

allow the order to be reduced by k – units.

Let's introduce a new unknown function by putting. Then $y^{(k)} = z$

$$y^{(k+1)} = z', \dots, y^{(k+2)} = z'', y^{(n)} = z^{(n-k)}$$

and equation

$$F(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}) = 0$$

will be rewritten as:

$$F(x, z, z', \dots, z^{(n-k)}) = 0$$

This is a differential equation of order $n - k < n$ with respect to the unknown function *z*. Substitution $y^{(k)} = z$ lowers the order of this equation by k units.

Let us assume that the resulting $(n-k)$ th order equation is integrated with the relation

$$z = \psi(x, C_1, C_2, \dots, C_{n-k}),$$

where $C_1, C_2, \dots, C_{n-k}$ – are arbitrary constants, represents the set of its solutions.

Substituting the value instead of $zy^{(k)}$, we obtain to determine the set of solutions to this equation of the form:

$$y^{(k)} = \psi(x, C_1, C_2, \dots, C_{n-k})$$

Integrating it sequentially k times, we obtain a set of solutions to this equation:

$$y = \varphi(x, C_1, C_2, \dots, C_n).$$

Example. Given an equation $y''' = \frac{y''}{x}$. Determine its solutions. Since the equation $y''' = \frac{y''}{x}$ does not contain y and y', the order can be lowered by putting. In this case $y'' = z$, the equation $y''' = z'$ will be written in the form $z' = \frac{z}{x}$. Solving it by the method of separation of variables, we obtain:

$$\frac{dz}{z} = \frac{dx}{x},\ ln|z| = ln|x| + ln|C_1|,\ z = C_1 x,\ C_1 \neq 0$$

By adding the solution $z = 0$, lost when separating the variables, we find the set of all solutions to the auxiliary equation in the form:

$$z = C_1 x,$$

where C_1 – is an arbitrary constant.

Substituting y'' instead of $z = C_1 x$ into the relation to determine y, we have a second-order equation:

$$y'' = C_1 x$$

Integrating it we find:

$$y' = C_1 \frac{x^2}{2} + C_2,$$

$$y = C_1 \frac{x^3}{3!} + C_2 x + C_3.$$

This relationship will be a general solution to this equation in the regions

$$\{x > 0,\ -\infty < y < +\infty\}\ and\ \{x < 0,\ -\infty < y < +\infty\}$$

c) Equations that do not contain an explicitly independent variable, equations of the $F(y, y', \dots, y^{(n)}) = 0$, form allow a decrease in order by one, if a new independent variable is taken, and the new desired function is taken $y' = p$. Applying the rule of differentiation of complex functions, we obtain:

$$y' = p, \qquad y'' = \frac{dy'}{dx} = \frac{dy'}{dy}\cdot\frac{dy}{dx} = \frac{dp}{dy}\cdot p$$

$$y''' = \frac{dy''}{dx} = \frac{dy''}{dy}\cdot\frac{dy}{dx} = \frac{dy''}{dy}\cdot p = \frac{d\left(\frac{dp}{dy}\cdot p\right)}{dy}\cdot p = \frac{d^2p}{dy^2}\cdot p^2 + \left(\frac{dp}{dy}\right)^2\cdot p$$

etc.

Each of the derivatives y of x the order m $(1 \leq m \leq n)$ is expressed in terms of the derivatives p of y the order and above $m - 1$. Substituting their values into this equation, we obtain a differential equation of order for the new unknown function $n - 1$:

$$F_1\left(y, p, \frac{dp}{dy}, \ldots, \frac{d^{n-1}p}{dy^{n-1}}\right) = 0$$

If this equation is integrated and

$$\Phi(y, p, C_1, C_2, \ldots, C_{n-1}) = 0$$

set of its solutions, then to find solutions to this differential equation it remains to solve the first-order equation:

$$\Phi(y, y', C_1, C_2, \ldots, C_{n-1}) = 0\,.$$

Example. Find solutions to the equation $yy'' - y'^2 - 4yy' = 0$. This is a second-order equation that does not contain explicit equations$x$. Let us denote by $p = y'$ the new required function, and we will consider y as a new independent variable. Then $y'' = \frac{dp}{dy}\cdot p$. With the new variables, this equation looks like:

$$py\frac{dp}{dy} - p^2 - 4yp = 0$$

$$p\left(y\frac{dp}{dy} - p - 4y\right) = 0$$

The equation

$$y\frac{dp}{dy} - p - 4y = 0$$

$$\frac{dp}{dy} = 4 + \frac{p}{y}, \qquad y \neq 0$$

homogeneous; assuming $p = uy$, we find:

$$u + \frac{du}{dy} \cdot y = 4 + u$$

$$du = \frac{4dy}{y}$$

$$u = 4ln|y| + 4ln|C_1|$$

$$p = 4yln|C_1y|.$$

where C_1 − is an arbitrary, non-zero constant. By setting $p = \frac{dy}{dx}$, to determine the solutions to this equation, we obtain the relation

$$\frac{dy}{dx} = 4yln|C_1y|.$$

Integrating it, we find a family of solutions to this equation:

$$ln|ln|C_1y|| = 4x + C_2.$$

Where C_1 *and* C_2 − are arbitrary constants.

Since the set of solutions to the equation

$$py\frac{dp}{dy} - p^2 - 4py = 0$$

consists of solutions to equations

$$y\frac{dp}{dy} - p - 4y = 0 \ and \ p = 0$$

To the found family of solutions to this equation, we should add solutions to the equation $y' = 0$, that is, $y = C$.

4 - §. Linear differential equations of higher order

Differential equation of nth order

$$F(x,y,y',\dots,y^{(n)}) = 0$$

is called linear if the function

$$F(x,y,y',\dots,y^{(n)})$$

linear with respect to the desired function y and all its derivatives. [9].

Any linear differential equation of nth order can be written as:

$$A_0(x)y^{(n)} + A_1(x)y^{(n-1)} + \dots + A_n(x)y + A_{n+1}(x) = 0 \qquad (1)$$

If the coefficients $A_0(x)$, $A_1(x)$, ... , $A_n(x)$ are constant, equation (1) is called a linear equation with constant coefficients. The free $A_{n+1}(x)$ term can be either constant or dependent on$x$.

Linear differential equations are accepted in the so-called reduced form:

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y = q(x) \qquad (2)$$

To move from equation (1) to equation (2), it is enough to divide both sides of equation (1) $A_n(x)$ and designate

$$p_i(x) = \frac{A_i(x)}{A_0(x)}, \quad q(x) = -\frac{A_{n+1}(x)}{A_0(x)}$$

Equations (1) and (2) are equivalent, where $A_0(x) \neq 0$.

An equation of the form (2) with continuous coefficients $p_1(x), p_2(x), \dots, p_n(x)$ and the right-hand side q(x) over some interval (a,b) of the OX axis (finite or infinite).

Under such assumptions, equation (2) in the region

$\{a < x < b, -\infty < y < +\infty, -\infty < y' < +\infty, \dots, -\infty < y^{(n-1)} < +\infty\}$

(n + 1) – dimensional space satisfies the conditions of the theorem of existence and uniqueness of solution.

Equation (2), resolved with respect to the highest derivative, has the form:

$$y^{(n)} = -p_1(x)y^{(n-1)} - \dots - p_n(x)y + q(x)$$

function

$$f(x,y,y',\dots,y^{(n-1)}) = -p_1(x)y^{(n-1)} - \dots - p_n(x)y + q(x)$$

is continuous in this domain and has continuous partial derivatives with respect to $y, y', \dots, y^{(n-1)}$:

$$\frac{\partial f}{\partial y} = -p_n(x), \quad \frac{\partial f}{\partial y'} = -p_{n-1}(x), \ldots, \frac{\partial f}{\partial y^{(n-1)}} = -p_1(x).$$

Any system of initial data

$$x_0, y_0, y'_0, \ldots, y_0^{(n-1)},$$

where $a < x_0 < b$, $y_0, y'_0, \ldots, y_0^{(n-1)}$ – any numbers, determines, in a certain neighborhood of points x_0, a unique solution to equation (2). This solution will be determined not only in a neighborhood of points x_0, but throughout the entire interval (a, b).

Equation (2) is called a linear inhomogeneous equation or a linear equation with the right-hand side if the function q(x) is not identically zero. If then $q(x) \equiv 0$, equation (2) is called a linear homogeneous equation or a linear equation without a right-hand side.

4 - §.Linear homogeneous differential equations with arbitrary coefficients.

Consider the linear homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0 \qquad (1)$$

with continuous coefficients in the interval (a, b). [9].

Each point of the regionσ, which is a strip $\{a < x < b - \infty < y < +\infty\}$, passes through a solution to equation (1), and this solution is uniquely determined by specifying its initial conditions. Let us denote by $L(y)$ the result of applying to the function$y$ the set of operations indicated by the left side of equation (1):

$$L(y) = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y \qquad (2)$$

and we will call $L(y)$ a linear differential operator.

The linear differential operator $L(y)$ assigns each n differentiable function to some function x.

For example, if

$$L(y) = y'' - 5xy' + x^2y, \qquad \text{then}$$

$$L(e^{2x}) = 4e^{2x} - 10xe^{2x} + x^2e^{2x}$$

$$L(x^5) = 20x^3 - 25x^5 + x^7$$

The linear differential operator (2) has the following properties.

1) If there y is an n differentiable function and C -is any number, then

$$L(Cy) = CL(y).$$

Really,

$$L(Cy) = (Cy)^{(n)} + p_1(x)(Cy)^{(n-1)} + \cdots + p_n(x)(Cy) =$$

$$= Cy^{(n)} + p_1(x)Cy^{(n-1)} + \cdots + p_n(x)Cy =$$

$$= C[y^{(n)} + p_1y^{(n-1)} + \cdots + p_n(x)y] = CL(y)$$

2) If y_1 and y_2 are n - differentiable functions, then

$$L(y_1 + y_2) = L(y_1) + L(y_2).$$

Really,

$$L(y_1 + y_2) = (y_1 + y_2)^2 + p_1(x)(y_1 + y_2)^{(n-1)} + \cdots + p_n(x)(y_1 + y_2) =$$

$$=[y_1^{(n)} + p_1(x)y_1^{(n-1)} + \cdots + p_n(x)y_1] + [y_2^{(n)} + p_1(x)y_2^{(n-1)} + \cdots + p_n(x)y_2] =$$

$$= L(y_1) + L(y_2)$$

Solutions of a linear homogeneous equation have the following properties:

1. If function y1 is a solution to equation (1), then function Cy_1, where C - is any number, is also its solution.
2. If the functions y_1 and y_2 are solutions to equation (1), then the function $y_1 + y_2$ is also its solution.

These properties are a direct consequence of the properties of the linear operator $L(y)$.

Based on properties 1 and 2, we conclude that if $y_1, y_2, \dots, y_n$-n any solutions of equation (1), then their linear combination

$$C_1y_1 + C_2y_2 + \cdots + C_ny_n$$

with arbitrary constant coefficients $C_1, C_2, \dots, C_n$ is also a solution to this equation.

5 - §. Linear dependence or linear independence of functions. Determinant of Vronsky properties

Let us consider a system of n functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ defined on the same interval (a, b) of the OX axis. [9]

These functions are called linearly dependent on the interval (a, b) if there are numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ that are not all equal to zero, such that for all x intervals (a, b) the relation is identically satisfied

$$\alpha_1\varphi_1(x) + \alpha_2\varphi_2(x) + \cdots + \alpha_n\varphi_n(x) = 0 \qquad (1)$$

If the functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ are linearly dependent on the interval (a, b), then at least one of these functions is a linear combination of the others.

Indeed, in relation (1) there are coefficients different from zero. Let, for example $\alpha_n \neq 0$. Then the function $\varphi_n(x)$ is a linear combination of the remaining functions of the system:

$$\varphi_n(x) = \beta_1 \varphi_1(x) + \beta_2 \varphi_2(x) + \cdots + \beta_{n-1} \varphi_{n-1}(x), \ where \ \beta = -\frac{\alpha_i}{\alpha_n},$$

$$i = 1, 2, \ldots, n-1$$

Example 1. Function $\varphi_1(x) = sin^2x, \ \varphi_2(x) = cos^2x, \ \varphi_3(x) \equiv 1$ are linearly dependent on any interval (a, b). Indeed, assuming $\alpha_1 = 1, \ \alpha_2 = 1, \alpha_3 = -1$, we obtain, based on the well – known trigonometric identity, that

$$1 \cdot sin^2x + 1 \cdot cos^2x + (-1) \cdot 1 \equiv 0$$
$$sin^2x + cos^2x - 1 \equiv 0 \ 1$$
$$-1 \equiv 0$$

Example 2. Functions

$\varphi_1(x) = sin^2x, \ \varphi_2(x) = x, \ \varphi_3(x) = cos^2x, \ \varphi_4(x) = 1, \ \varphi_5(x) = e^x$ linearly dependent on any interval (a, b), assuming $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 1, \alpha_4 = -1, \ \alpha_5 = 0$ we get:

$$1 \cdot sin^2x + 0 \cdot x + 1 \cdot cos^2x + (-1) \cdot 1 + 0 \cdot e^x \equiv 0.$$

$$sin^2x + 0 + cos^2x - 1 + 0 \equiv 0$$
$$1 - 1 \equiv 0$$

Example 3. Functions

$$\varphi_1(x) = \sqrt{x}, \ \varphi_2(x) = \frac{1}{x}, \ \varphi_3(x) \equiv 0, \ \varphi_4(x) = x^2$$

on the interval $0 < x < 1$ are linearly dependent, assuming $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 2, \ \alpha_4 = 0$, we obtain:

$$0 \cdot \sqrt[4]{x} + 0 \cdot \frac{1}{x} + 2 \cdot 0 + 0 \cdot x^2 \equiv 0$$
$$0 \equiv 0$$

The linear dependence of two unequal identically zero functions on the interval (a, b) is equivalent to the proportionality of these functions $\varphi_1(x)$ *and* $\varphi_2(x)$.

Indeed, if

$$\alpha_1\varphi_1(x) + \alpha_2\varphi_2(x) \equiv 0$$

and

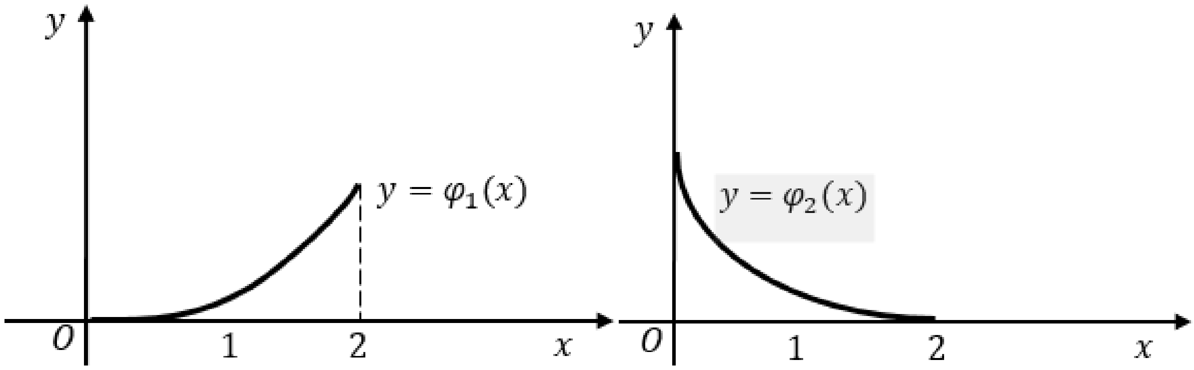
$$\alpha_1 \neq 0, \ \textit{that} \ \varphi_1(x) \equiv -\frac{\alpha_2}{\alpha_1}\varphi_2(x).$$

Functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ defined on the interval (a, b) of the OX axis are called linearly independent on this interval if from the relation

$$\alpha_1\varphi_1(x) + \alpha_2\varphi_2(x) + \dots + \alpha_n\varphi_n(x) \equiv 0$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ - are the numbers , that $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

If the functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ are linearly independent on the interval (a, b), then none of them is a linear combination of the others.



Example. Functions $\varphi_0(x) \equiv 1$, $\varphi_1(x) = x$, $\varphi_2(x) = x^2, \dots, \varphi_n(x) = x^n$, where n - is any natural number, linearly independent on the entire number axis.

Indeed, if you make a linear combination of these functions with coefficients $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$, you get a polynomial:

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n$$

A polynomial of degree not greater than *n* cannot have more than *n* real roots. Therefore, the identity equality

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n \equiv 0$$

perhaps only if

$$\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$$

Let's consider another example of linearly independent functions on the interval (0, 2). Let

$$\varphi_1(x) = \begin{cases} 0 & 0 < x < 1, \\ (x-1)^4, & 1 \le x < 2, \end{cases} \quad \varphi_2(x) = \begin{cases} (x-1)^4, & 0 < x < 1 \\ 0, & 1 \le x < 2 \end{cases}$$

From this system is shown in Figure a) and b).
If for any value x from the interval (0, 2) the equality holds

$$\alpha_1\varphi_1(x) + \alpha_2\varphi_2(x) \equiv 0$$

then, substituting $x = \frac{1}{2}$, we get that

$$\varphi_1\left(\frac{1}{2}\right) = 0, \ \varphi_2\left(\frac{1}{2}\right) = \frac{1}{16}$$

$$\alpha_1 \cdot 0 + \alpha_2 \cdot \frac{1}{16} = 0, \text{ that is } \alpha_2 = 0.$$

Substituting then $x = \frac{3}{2}$, we get that

$$\varphi_1\left(\frac{3}{2}\right) = \frac{1}{16}, \quad \varphi_2\left(\frac{3}{2}\right) = 0,$$

$$\alpha_1 \cdot \frac{1}{16} + 0 \cdot 0 = 0, \text{ that is } \alpha_1 = 0.$$

Thus, from the identical equality

$$\alpha_1\varphi_1(x) + \alpha_2\varphi_2(x) \equiv 0$$

it follows that $\alpha_1 = \alpha_2 = 0$. This means that the functions $\varphi_1(x)$ *and* $\varphi_2(x)$ are linearly independent.

To study some systems of functions, linear dependence was proposed by the Polish mathematician Jozef Wronski.

Theorem 1. If y_1 and y_2 are two particular solutions of a linear homogeneous second-order equation

$$y'' + a_1y' + a_2y = 0 \qquad (3)$$

then $y_1 + y_2$ also has a solution to this equation.

Proof. Since y_1 and y_2 are solutions to the equation, then

$$y_1'' + a_1 y_1' + a_2 y_1 = 0, \qquad y_2'' + a_1 y_2' + a_2 y_2 = 0 \quad (4)$$

Substituting the sum $y_1 + y_2$ into equation (3) and taking into account identities (4), we will have

$$(y_1 + y_2)'' + a_1(y_1 + y_2)' + a_2(y_1 + y_2) = (y_1'' + a_1 y_1' + a_2 y_1) + \\ +(y_2'' + a_1 y_2' + a_2 y_2) = 0 + 0 = 0$$

that is, $y_1 + y_2$ has a solution to the equation.

Theorem 2. If y_1 is a solution to equation (3) and C is a constant, then Cy_1 is also a solution to equation (3).

Proof. Substituting the expression Cy_1 into equation (3), we obtain

$$(Cy_1)'' + a_1(Cy_1)' + a_2(Cy_1) = C(y_1'' + a_1 y_1' + a_2 y_1) = C \cdot 0 = 0$$

Thus the theorem is proven.

Definition 2. Two solutions of equation (3) y_1 *and* y_2 are called linearly independent in the segment $[a, b]$ if their ratio in this segment is not constant, that is, if

$$\frac{y_1}{y_2} \neq const$$

Otherwise, the solutions are called linearly dependent. In other words, two solutions y_1 *and* y_2 are called linearly dependent on the interval $[a, b]$ if such a constant number exists λ, that in $\frac{y_1}{y_2} = \lambda$ *at* $a \leq x \leq b$. This case $y_1 = \lambda y_2$.

Definition3. If y_1 *and* y_2 is the essence of the function from *x,* then the determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

is called the Wronski definition or the Livronskian definition of given functions.

Theorem 3. If the functions y_1 *and* y_2 are linearly independent on the interval $[a, b]$, then the Wronski determinant on this interval is identically equal to zero.

Indeed, if $y_2 = \lambda y_1$, where $\lambda = const$, and $y_2' = \lambda y_1'$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & \lambda y_1 \\ y_1' & \lambda y_1' \end{vmatrix} = \lambda \begin{vmatrix} y_1 & y_1 \\ y_1' & y_1' \end{vmatrix} = 0$$

Example 1. Given the equation $y''' + \frac{2}{x}y'' - y' + \frac{1}{xlnx}y = x$ and a known particular solution $y_1 = lnx$ of the corresponding homogeneous equation. Reduce the order of the equation.

Solution. Let's use the substitution $y = lnx \int zdz,$ $where\ z -$ new unknown function. Then, substituting the corresponding derivatives

$$y' = \frac{1}{x}\int zdz + zlnx, \quad y'' = -\frac{1}{x^2}\int zdx + \frac{2z}{x} + z'lnx$$

$$y''' = \frac{2}{x^3}\int zdx - \frac{3z}{x^2} + \frac{3z'}{x} + z''lnx$$

into this equation, we obtain a second-order equation

$$z''lnx + \frac{2lnx}{3} \cdot z' + \left(\frac{1}{x^2} - lnx\right) z = x$$

Let the $y_1, y_2, \ldots, y_n - n$ functionsbe defined and be differentiated n – 1 times on the interval (a, b).

Determinant of n th order

$$W = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y'_1 & y'_2 & \ldots & y'_n \\ \ldots & \ldots\ldots. & & \ldots. \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}$$

is called the Wronski determinant, or Liveronskian, for these functions. The Wronski determinant is also a function of *x* defined on the interval (a, b):

$$W = W(x).$$

If you put it on the matrix, this will turn out $y_1 = sinx,$ $y_2 = e^{-x},$ $y_3 = x^2,$

$$W = \begin{vmatrix} sinx & e^{-x} & x^2 \\ cosx & -e^{-x} & 2x \\ -sinx & e^{-x} & 2 \end{vmatrix}$$

Theorem 4. If the functions $y_1, y_2, \ldots, y_n - n$ are linearly dependent on the interval (a, b), then the Wronski determinant compiled for them on this interval is identically equal to zero.

Proof. Since the functions $y_1, y_2, \dots, y_n$ are linearly dependent on the interval (a, b), then at least one of these functions, let it be y_n , is a linear combination of the remaining functions:

$$y_n = \beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_{n-1} y_{n-1}$$

where $\beta_1, \beta_2, \dots, \beta_{n-1}$- is some number.

Differentiating this identity successively n–1 times, we obtain:

$$y'_n = \beta_1 y'_1 + \beta_2 y'_2 + \cdots + \beta_{n-1} y'_{n-1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y_n^{(n-1)} = \beta_1 y_1^{(n-1)} + \beta_2 y_2^{(n-1)} + \cdots + \beta_{n-1} y_{n-1}^{(n-1)}$$

The Wronski determinant corresponding to this system of functions will be written in the form:

$$W(x) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y'_1 & y'_2 & \dots & y'_n \\ \dots & \dots. & & \dots. \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} =$$

$$= \begin{vmatrix} y_1 & y_2 & \dots & \beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_{n-1} y_{n-1} \\ y'_1 & y'_2 & \dots & \beta_1 y'_1 + \beta_2 y'_2 + \cdots + \beta_{n-1} y'_{n-1} \\ \dots.. & \dots. & & \dots. \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & \beta_1 y_1^{(n-1)} + \beta_2 y_2^{(n-1)} + \cdots + \beta_{n-1} y_{n-1}^{(n-1)} \end{vmatrix}$$

Subtracting from the elements of the last column of the determinant the corresponding elements of the first column, multiplied byβ_1, then the elements of the second column, multiplied by β_2, etc., elements $(n - 1)$ of the last column, multiplied by β_{n-1}, we obtain that

$$W(x) = \begin{vmatrix} y_1 & y_2 & \dots & y_{n-1} & 0 \\ y'_1 & y'_2 & \dots & y'_{n-1} & 0 \\ \dots & \dots. & & \dots. & \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_{n-1}^{(n-1)} & 0 \end{vmatrix} \equiv 0$$

Theorem 5. If are $y_1, y_2, \dots, y_n - n$ linearly independent on the interval (a, b) solutions of a linear homogeneous equation of n th order $L(y) = 0^2$, then the Wronski determinant compiled for them at the first point of the interval (a, b) is not equal to zero.

Proof. Let's assume the opposite. Let us assume that there is a point at which x_0, $a < x_0 < b$ the Wronski determinant, compiled for the functions $y_1, y_2, \ldots, y_n$, is equal to zero:

$$W(x) = \begin{vmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y_n'(x_0) \\ \ldots & \ldots. & & \ldots. \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{vmatrix} = 0$$

Let us consider an auxiliary system of n linear homogeneous algebraic equations with unknowns $\alpha_1, \alpha_2, \ldots, \alpha_n$:

$$\begin{gathered} \alpha_1 y_1(x_0) + \alpha_2 y_2(x_0) + \cdots + \alpha_1 y_n(x_0) = 0, \\ \alpha_1 y_1'(x_0) + \alpha_2 y_2'(x_0) + \cdots + \alpha_n y_n'(x_0) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_1 y_1^{(n-1)}(x_0) + \alpha_2 y_2^{(n-1)}(x_0) + \cdots + \alpha_n y_n^{(n-1)}(x_0) = 0 \end{gathered} \qquad (2)$$

This linear homogeneous system of equations has a non - zero solution, since the determinant of the system $W(x_0)$ (the determinant of the coefficients of the unknowns) is equal to zero.

Let us denote by, $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n$ - the non - zero solution of system (2) and consider the function

$$\tilde{y} = \tilde{\alpha}_1 y_1 + \tilde{\alpha}_2 y_2 + \cdots + \tilde{\alpha}_n y_n$$

This function, being a linear combination of solutions to the equation $L(y) = 0$ will itself be a solution to the same equation. Because

$$\begin{gathered} \tilde{y} = \tilde{\alpha}_1 y_1 + \tilde{\alpha}_2 y_2 + \cdots + \tilde{\alpha}_n y_n \\ \tilde{y}' = \tilde{\alpha}_1 y_1' + \tilde{\alpha}_2 y_2' + \cdots + \tilde{\alpha}_n y_n' \\ \ldots\ldots\ldots\ldots\ldots \\ \tilde{y}^{(n-1)} = \tilde{\alpha}_1 y_1^{(n-1)} + \tilde{\alpha}_2 y_2^{(n-1)} + \cdots + \tilde{\alpha}_n y_n^{(n-1)} \end{gathered}$$

that at $x = x_0$ we have equations (2):

$$\begin{gathered} \tilde{y}(x_0) = \tilde{\alpha}_1 y_1(x_0) + \tilde{\alpha}_2 y_2(x_0) + \cdots + \tilde{\alpha}_n y_n(x_0) = 0 \\ \tilde{y}'(x_0) = \tilde{\alpha}_1 y_1'(x_0) + \tilde{\alpha}_2 y_2'(x_0) + \cdots + \tilde{\alpha}_n y_n'(x_0) = 0 \\ \ldots\ldots\ldots\ldots\ldots \\ \tilde{y}^{(n-1)}(x_0) = \tilde{\alpha}_1 y_1^{(n-1)}(x_0) + \tilde{\alpha}_2 y_2^{(n-1)}(x_0) + \cdots + \tilde{\alpha}_n y_n^{(n-1)}(x_0) = 0 \end{gathered}$$

This means that the solution $\tilde{y}$ to the equation $L(y) = 0$ satisfies the initial conditions $x_0, 0, 0, \ldots, 0$.

Every linear homogeneous equation has a so-called trivial solution, identically equal to zero: $y = 0$. This solution also satisfies the initial conditions $x_0, 0, 0, \ldots, 0$.

By the theorem of existence and uniqueness of the solution of the equation $L(y) = 0$ specifying the initial data system uniquely determines the solution, that is, the solution must coincide with the solution identically $\tilde{y}$ equal to zero:

$$\tilde{y} \equiv 0, \text{ or, what's more, } \tilde{\alpha}_1 y_1 + \tilde{\alpha}_2 y_2 + \cdots + \tilde{\alpha}_n y_n \equiv 0$$

Since among the numbers, $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n$ there are different from zero, it follows from the last relation that the functions $y_1, y_2, \ldots, y_n$ interval (a, b) are linearly dependent, which contradicts the condition. The assumption is that the determinant $W(x)$ can vanish on the interval (a, b) is eliminated.

Example. As was established in the example above, the functions

$$\varphi_0(x) \equiv 1, \ \varphi_1(x) = x, \ \varphi_2(x) = x^2, \ldots, \varphi_n(x) = x^n.$$

where n is any natural number, linearly independent on the entire number axis.

Indeed, if you make a linear combination of these functions with coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$, you get a polynomial:

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n$$

A polynomial of degree not greater than n cannot have more than n real roots. Therefore, the identity equality

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n \equiv 0$$

perhaps only if $\alpha_0 = \alpha_1 = \ldots = \alpha_n = 0$

Let's consider another example of linearly independent functions on the interval (0, 2). Let

$$\varphi_1(x) = \begin{cases} 0 & 0 < x < 1, \\ (x-1)^4, & 1 \le x < 2, \end{cases} \qquad \varphi_2(x) = \begin{cases} (x-1)^4, & 0 < x < 1 \\ 0, & 1 \le x < 2 \end{cases}$$

are linearly independent on the interval (0, 2). Wherein

$$\varphi_1'(x) = \begin{cases} 0 & 0 < x < 1 \\ 4(x-1)^3, & 1 \le x < 2 \end{cases}, \qquad \varphi_2'(x) = \begin{cases} 4(x-1)^3, & 0 < x < 1 \\ 0, & 1 \le x < 2 \end{cases}$$

Let's compose a Vronsky determinant for them:

$$W(x) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix}$$

For anyone $x, 0 < x < 1$, we have:

$$W(x) = \begin{vmatrix} 0 & (x-1)^4 \\ 0 & 4(x-1)^3 \end{vmatrix} = 0$$

For any one $x,\ 1 \le x < 2$, we also get that

$$W(x) = \begin{vmatrix} (x-1)^4 & 0 \\ 4(x-1)^3 & 0 \end{vmatrix} = 0$$

that is, on the interval (0, 2) $W(x) \equiv 0$.

The property of the Wronski determinant follows that if $y_1, y_2, \dots, y_n$ are solutions of the linear homogeneous equation $L(y) = 0$, defined on the interval (a, b), then the Wronski determinant constructed for them is either identically equal to zero on the interval (a, b), or not equal to zero at the first point of the interval (a, b).

Try to decide for yourself [3]

1. Integrate the equation $y'' + \frac{2}{x}y' + y = 0,$, which has a particular solution $y_1 = \frac{sinx}{x}$.

2. Reduce the order and integrate the equation $y''sin2x = 2y$, which has a particular solution $y = ctgx$.

3. The equation $y'' - \frac{y'}{x} + \frac{y}{x} = 0$ has a particular solution $y = x$. Lower the order and integrate this equation.

4. The equation $y'' + (tgx - 2\,ctgx)\,y' + 2ctg2x \cdot y = 0$ has a particular solution $y = sinx$. Lower the order and integrate this equation.

Answers.

1) $y = \frac{sinx}{x} \int \frac{C_1 dx}{sin^2x} = \frac{sinx}{x}(C_2 - C_1 ctgx) = C_2 \cdot \frac{sinx}{x} - C_1 \frac{cosx}{x}$

2) $y = C_2 + (C_1 - C_2 x)ctgx$

3) $y = -\frac{1}{2}xln^2 x + C_1 xlnx + C_2 x$

4) $y = C_1 sinx + C_2 sin^2x$

6 - §. Characteristic equation

1) **Characteristic equation**.
Differential equation

$$y'' + py' + qy = 0 \qquad (1)$$

where p and q are constant numbers. To understand the essence of the matter, let's start with an example [7]

$$y'' - 5y' + 6y = 0 \qquad (2)$$

The solution to this differential equation must be a function that, when substituted into the equation, transforms its identity. The zero part of the equation is the sum of the function itself and its derivatives y' and y'' taken with some constant coefficients. For such a sum to turn out to be identically zero $y,\ y'$ and y'', they must be similar to each other. Therefore, for example, none of the functions

$$y = x^3, \quad y = tgx, \quad y = lnx$$

obviously cannot be a solution to equation (2). The solution to differential equation (2) will be the function $y = e^x$. Substituting it into the equation, we are immediately convinced that it is not a solution. But not only, but each $y = e^x$ of the functions $y = e^{2x}$, $y = e^{3x}$, $y = e^{-x}$, ... will also be similar to its derivatives. To avoid an infinite number of trials, consider the function

$$y = e^{kx} \qquad (3)$$

and we will try to select k so that this function satisfies (2). Because

$$y' = ke^{kx}, \qquad y'' = k^2e^{kx}$$

then, substituting (3) to the left side of (2), we get

$$k^2e^{kx} - 5ke^{kx} + 6e^{kx}$$

or, which is the same thing,

$$e^{kx}(k^2 - 5k + 6)$$

For this expression to be zero, it must be

$$k^2 - 5k + 6 = 0 \qquad (4)$$

We will find the required k by solving equation (4). The root equations (4) are

$$k_1 = 2, \quad k_2 = 3$$

Thus, we have found even two of the values we need$k$. In accordance with these names, two solutions have been found

$$y_1 = e^{2x}, \ y_2 = e^{3x}$$

our differential equation. These solutions are linearly independent, since

$$\frac{y_2}{y_1} = e^x \neq const$$

These solutions make it possible to construct a general solution to equation (2), namely

$$y = C_1e^{2x} + C_2e^{3x}$$

Let us now move on to consider the general case of differential equation (1). Here, too, the desired function must be similar to its derivatives y' and y''. Therefore, it is natural here to engage in the selection of such that the function

$$y = e^{kx}$$

turned out to be a solution to differential equation (1). Substituting this function to the left side of (1) gives the expression

$$e^{kx}(k^2 + pk + q)$$

For this expression to be zero, k must be the root of the quadratic equation

$$k^2 + pk + q = 0 \qquad (5)$$

which is called the characteristic equation for the differential equation (1).
When solving the quadratic equation (5), 3 cases may occur:

I. Roots (5) real and various
II. Roots (5) real and equal
III. The roots (5) are imaginary.

Let the roots (5) be real numbers

$$k_1 = a,\ k_2 = b,\ (a \neq b)$$

Then (1) has 2 solutions

$$y_1 = e^{ax},\ y_2 = e^{bx}$$

and due to their linear independence, since

$$\frac{e^{ax}}{e^{bx}} \neq const$$

the general solution (1) is:

$$y = C_1e^{ax} + C_2e^{bx} \qquad (6)$$

Examples.1) Characteristic equation here $y'' - 12y' + 35y = 0.$

$$k^2 - 12k + 35 = 0$$

Its roots are because the general solution of the differential equation
$k_1 = 5,\ k_2 = 7$

$$y = C_1e^{5x} + C_2e^{7x}$$

2) $y'' - 16y = 0.$ Here is the characteristic equation $k^2 - 16 = 0$. His roots are a general solution $k_{1,2} = \pm 4$

$$y = C_1e^{4x} + C_2e^{-4x}$$

3) $y'' - 2y' = 0.$Here the characteristic equation has the form. His roots are because $k^2 - 2k = 0$, $k_1 = 0,\ k_2 = 2$

$$y = C_1 + C_2e^{2x}$$

4) $y''' - 3y'' + 2y' = 0.$ Here the characteristic equation has the form

$$k^3 - 3k^2 + 2k = 0 \quad \text{or} \quad k(k^2 - 3k + 2) = 0$$

It has roots

$$k_1 = 0,\ k_2 = 1,\ k_3 = 2.$$

Means,

$$y = C_1 + C_2e^x + C_3e^{2x}$$

The general solution of a 3rd order differential equation depends on three arbitrary constants.

5) $y^{(4)} - 29y'' + 100y = 0.$ Here is the characteristic equation

$$k^4 - 29k^2 + 100 = 0$$

that is, it is a biquadratic equation. Believing $k^2 = z$, we find

$$z^2 - 29z + 100 = 0$$

where $z_1 = 4, z_2 = 25.$

But then $k_{1,2} = \pm 2,\ k_{3,4} = \pm 5$ and

$$y = C_1e^{2x} + C_2e^{-2x} + C_3e^{5x} + C_4e^{-5x}$$

2) **The case of equal roots of the characteristic equation.**
For the equation

$$y'' - 6y' + 9y = 0 \quad (1)$$

The characteristic equation has the form

$$k^2 - 6k + 9 = 0 \qquad (2)$$

This quadratic equation has only one root $k = 3$. Therefore, our theory gives only a code for a particular solution

$$y_1 = e^{3x} \qquad (3)$$

Differential equation (1), but this is not enough to construct its general solution. Equation (2) has more than one, but two equal roots $k_1 = 3$ *and* $k_2 = 3$, but here it turns out to be just a turn of phrase that doesn't save anything. Indeed, if instead of one solution (3) we consider two of them

$$y_1 = e^{3x}, \qquad y_2 = e^{3x} \qquad (4)$$

and suppose $y = C_1e^{3x} + C_2e^{3x}$

then yall will not be a general solution (1), because

$$y = (C_1 + C_2)e^{3x} = Ce^{3x} \quad (C = C_1 + C_2),$$

that is, y depends only on one arbitrary constant. This is completely unnatural, the "two" solutions (4) are linearly independent. Solving equation (1) and function

$$y_1 = xe^{3x} \qquad (5)$$

This is done by simple substitution to the y_2 left side of (1). Because

$$y_2' = e^{3x} + 3xe^{3x}, \qquad y_2'' = 6e^{3x} + 9xe^{3x}$$

and this is identically equal to zero. This means that (5) is also a solution to differential equation (1), and the linear independence of functions (3) and (5) is obvious. Notthenfunction

$$y = C_1e^{3x} + C_2xe^{3x}$$

will be a general solution to equation (1).

Let us now consider the cases of equal roots of the characteristic equation in general form.

For the equation

$$y'' + py' + qy = 0 \qquad (6)$$

characteristic serves as equation

$$k^2 + pk + q = 0 \qquad (7)$$

His roots

$$k_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

The condition for the coincidence of these roots is equality

$$\frac{p^2}{4} = q \qquad (8)$$

Let this equality be fair. Then (7) has only one root

$$k_1 = -\frac{p}{2}$$

leading to resolution

$$y_1 = e^{-\frac{p}{2}x}$$

differential equation (6). Let us make sure that alongy_1 with solution (6) there will be

$$y_2 = xe^{-\frac{p}{2}x}$$

Because

$$y_2' = e^{-\frac{p}{2}x} - \frac{p}{2}xe^{-\frac{p}{2}x}, \quad y_2'' = -pe^{-\frac{p}{2}x} + \frac{p^2}{4}xe^{-\frac{p}{2}x},$$

then the result of substituting y_2 to the left side of (6) has the form

$$-pe^{-\frac{p}{2}x} + \frac{p^2}{4}xe^{-\frac{p}{2}x} + p\left(-e^{-\frac{p}{2}x} - \frac{p}{2}xe^{-\frac{p}{2}x}\right) + qxe^{-\frac{p}{2}x}$$

or, which is the same thing,

$$\left(q - \frac{p^2}{4}\right) xe^{-\frac{p}{2}x} \qquad (9)$$

By virtue of (8), this expression is equal to zero, which proves the statement. This means that $y_1$andy_2 are two (obviously linearly independent) solutions of (6) and the general solution of this differential equation

$$y = C_1 e^{-\frac{p}{2}x} + C_2 x e^{-\frac{p}{2}x}.$$

Theorem. If the characteristic equation (7) has only one root

$$k_1 = a,$$

tone along with function

$$y_1 = e^{ax}$$

the solution to equation (6) will be the function

$$y_2 = xe^{ax} \qquad (10)$$

Note that when, in addition to the root $k_1 = a$, the characteristic equation (7) has a root $k_1 = b$ different from it, then function (10) will not be a solution to (7). Indeed, substitution of (10) to the left side of (6) gives (9), and this expression is identically equal to zero only under condition (8), that is, under the condition that the roots of equation (7) are equal.

Examples. 1) Here is the characteristic equation $y'' - 10y' + 25y = 0.$

$$k^2 - 10k + 25 = 0$$

The only root of this equation $k_1 = 5$. The general solution to the differential equation is:

$$y = C_1 e^{5x} + C_2 x e^{5x}$$

2) $y'' = 0$. Here the characteristic equation is: $k^2 = 0$. It has only one root $k_1 = 0$. This means that the general solution of the differential equation

$$y = C_1 + C_2 x$$

3) Given an equation. Write a characteristic equation $y'' - 4y' + 4y = 0$

$$k^2 - 4k + 4 = 0.$$

We find its roots:

$$k_1 = k_2 = 2.$$

The general integral will be

$$y = C_1 e^{2x} + C_2 x e^{2x}$$

7 - §. Inhomogeneous linear equations of the second order with constant coefficients

Let us have the equation

$$y'' + py' + qy = f(x) \qquad (1)$$

where p and q - are real numbers. [1].

I. Let the right side of equation (1) be the product of an exponential function and a polynomial, that is, it has the form

$$f(x) = P_n(x)\,e^{\alpha x} \qquad (2)$$

where P_n(x) is a polynomial of n th degree. Then the following special cases are possible:

a) Number $\boldsymbol{\alpha}$ is not a root of the characteristic equation

$$k^2 + pk + q = 0$$

In this case, a particular solution must be sought in the form

$$y^* = (A_0x^n + (3)A_1x^{n-1} + \cdots + A_n)e^{\alpha x} = Q_n(x)e^{\alpha x} \qquad (3)$$

Indeed, substituting y*into equation (1) and reducing all terms by a factor $e^{\alpha x}$, we will have:

$$Q''_n(x) + (2\alpha + p)Q'_n(x) + (\alpha^2 + p\alpha + q)Q_n(x) = P_n(x) \quad (4)$$

$Q_n(x)$- n polynomial of degree*,* n – 1– polynomial of degree , $Q'_n(x)$ $Q''_n(x)$- a polynomial of degree n – 2. Thus, to the left and to the right of the equal sign there are polynomials of the n th degree. Equating the coefficients at the same powers of *x,* we obtain a system of $n + 1$ equations for determining the unknown coefficients $A_0, A_1, A_2, \ldots, A_n$.

b) number $\boldsymbol{\alpha}$ is a simple (single) root of the characteristic equation.

If, in this case, we began to look for a particular solution in the form (3), then in equality (4) on the left we would get a polynomial of $(n - 1)$ degree, since the coefficient of $Q_n(x)$, that is, is equal to zero, and the polynomials

$$\alpha^2 + p\alpha + q$$

$Q'_n(x)$ *and* $Q''_n(x)$ have a degree less than *n*. There fore, under no circumstances

A_0, A_1, A_2, ... , A_n equality (4) would not be an identity. Therefore, in the case under consideration, a particular solution must be taken in the form of a polynomial of the $(n + 1)$ th degree, but without a free term:

$$y* = xQ_n(x)e^{\alpha x}$$

c) Number $\boldsymbol{\alpha}$ is a double root of the characteristic equation.
Then, as a result of substituting the function into the differential equation $Q_n(x)e^{\alpha x}$ the degree of the polynomial is reduced by two units. Indeed, if $\boldsymbol{\alpha}$ – rootcharacteristic equation $\alpha^2 + p\alpha + q = 0$, then; moreover, since$\boldsymbol{\alpha}$is a double root, then $2\alpha = -r.\ So,\ 2\alpha + p = 0.$

Consequently, the left side of equality (4) will remain, that is, a polynomial of $(n - 2)$ degree. In order to obtain a polynomial of degree n as a result of substitution, one should look for a particular solution in the form of a product $Q''_n(x)e^{\alpha x}$ to a polynomial of $(n + 2)$ degree. In this case, the free term of this polynomial and terms of the first degree will disappear during differentiation; therefore, they may not be included in a particular solution.

So, in the case when$\boldsymbol{\alpha}$is a double root of the characteristic equation, a particular solution can be taken in the form

$$y *= x^2 Q_n(x)e^{\alpha x}$$

Example 1. Find a general solution to the equation

$$y'' + 4y' + 3y = x$$

Solution. The general solution of the corresponding homogeneous equation is

$$\bar{y} = C_1 e^{-x} + C_2 e^{-3x}$$

Since the right side of this inhomogeneous equation has the form xe^{0x} (that is, the form $P_1(x)e^{0x}$, and 0 is not the root of the characteristic equation $k^2 + 4k + 3 = 0$, then we will look for a particular solution in the form $y^* = Q_1(x)e^{0x}$, that is, let's put

$$y = A_0 x + A_1$$

Substituting this expression into the given equation, we will have

$$4A_0 + 3(A_0 x + A_1) = x$$

Equating the coefficients at the same powers of x, we get

$$3A_0 = 1\ ,\quad 4A_0 + 3A_1 = 0$$

where

$$A_0 = \frac{1}{3},\quad A_1 = -\frac{4}{9}$$

Hence,

$$y^* = \frac{1}{3}x - \frac{4}{9}$$

Common decision

$$y = \bar{y} + y^*$$

will at

$$y = C_1e^{-x} + C_2 e^{-3x} + \frac{1}{3}x - \frac{4}{9}$$

Example 2. Find the general solution to the equation

$$y'' + 9y = (x^2 + 1)e^{3x}$$

Solution. We can easily find a general solution to the homogeneous equation:

$$\bar{y} = C_1 cos3x + C_2 sin3x$$

The right side of the given equation $(x^2 + 1)e^{3x}$ has the form $P_2(x)e^{3x}$. Since coefficient 3 in the exponent is the root of the characteristic equation, we look for a particular solution in the form

$$y* = Q_2(x)e^{3x} \quad \text{or} \quad y* = (Ax^2 + Bx + C)e^{3x}.$$

Substituting this expression into the differential equation, we will have

$$[9(Ax^2 + Bx + C) + 6(Ax + B) + 2A + 9(Ax^2 + Bx + C)]e^{3x} = (x^2 + 1)e^{3x}$$

Reducing by e^{3x} and equating the coefficients at the same powers of from x, we get

$$18A = 1, \qquad 12A + 18B = 0, \qquad 2A + 6B + 18C = 1,$$

where

$$A = \frac{1}{18}, \ B = -\frac{1}{27}, \ C = \frac{5}{81}$$

Therefore, the particular solution will be

$$y^* = (\frac{1}{18}x^2 - \frac{1}{27}x + \frac{5}{81})e^{3x}$$

and general solution

$$y = C_1 cos3x + C_2 sin3x + (\frac{1}{18}x^2 - \frac{1}{27}x + \frac{5}{81})e^{3x}$$

Example 3. Solve the equation $y'' - 7y' + 6y = (x - 2)e^{x}$.

Solution. Here the right-hand side has the form $P_1(x)e^{1x}$, and the coefficient 1 in the exponent is a simple root of the characteristic polynomial. Therefore, we look for a particular solution in the form

$$y^* = xQ_1(x)e^x \quad \text{or} \quad y^* = x(Ax + B)e^x$$

Substituting this expression into the equation, we have

$$[(Ax^2 + Bx) + (4Ax + 2B) + 2A - 7(Ax^2 + Bx) - 7(2Ax + B) + 6(Ax^2 + Bx)]e^x = (x - 2)e^x$$

or

$$(-10Ax - 5B + 2A)e^x = (x - 2)e^x$$

Equating the coefficients for the same powers of x, we get

$$-10A = 1, \quad -5B + 2A = -2$$

from where $A = -\frac{1}{10}$, $B = \frac{9}{25}$. There fore, the particular solution is

$$y^* = x\left(-\frac{1}{10}x + \frac{9}{25}\right)e^x$$

in general

$$y = C_1e^{6x} + C_2e^x + x\left(-\frac{1}{10}x + \frac{9}{25}\right)e^x$$

II. Let the right side have the form

$$f(x) = P(x)e^{\alpha x}\cos\beta x + Q(x)e^{\alpha x}\sin\beta x \qquad (5)$$

where $P(x)$ and $Q(x)$ - are polynomials.

This case can be considered using the technique used in the previous case, if we move from trigonometric functions to exponential ones. Replacing $\cos\beta x$ and $\sin\beta x$ th rough the exponential functions using Euler's formulas, we get

$$f(x) = P(x)e^{\alpha x}\frac{e^{i\beta x} + e^{-i\beta x}}{2} + Q(x)e^{\alpha x}\frac{e^{i\beta x} - e^{-i\beta x}}{2i}$$

or

$$f(x) = \left[\frac{1}{2}P(x) + \frac{1}{2i}Q(x)\right]e^{(\alpha+i\beta)x} + \left[\frac{1}{2}P(x) - \frac{1}{2i}Q(x)\right]e^{(\alpha-i\beta)x} \qquad (6)$$

Here in square brackets there are polynomials whose degrees are equal to the highest degree of the polynomials $P(x)$ and $Q(x)$. Thus, we obtained the right-hand side of the form considered in case 1.

So, if the right side of equation (1) has the form

$$f(x) = P(x)\, e^{\alpha x}\cos\beta x + Q(x)e^{\alpha x} \sin\beta x \qquad (7)$$

where $P(x)$ and $Q(x)$ - are polynomials in x, then the form of the particular solution is determined as follows:

a) if the number $\alpha + i\beta$ is not the root of the characteristic equation, then a particular solution to equation (1) should be sought in the form

$$y^* = U(x)e^{\alpha x}cos\beta x + V(x)e^{\alpha x}\, sin\beta x \qquad (8)$$

where $U(x)$ and $V(x)$ - are polynomials whose degree is equal to the highest degree of the polynomials $P(x)$ and $Q(x)$;

b) if the number is the root of the characteristic equation, then we look for a particular solution in the form $\alpha + i\beta$

$$y^* = x[U(x)e^{\alpha x}cos\beta x + V(x)e^{\alpha x}\, sin\beta x] \qquad (9)$$

Moreover, in order to avoid possible errors, it should be noted that the indicated forms of particular solutions (8) and (9), obviously, are preserved even in the case when on the right side of equation (1) one of the polynomials

P(x) and Q(x) are identically equal to zero, that is, when the right-hand side has the form $P(x)e^{\alpha x}cos\beta x$ or $Q(x)e^{\alpha x}sin\beta x$.

Let us next consider an important special case. Let the right-hand side of a second-order linear equation have the form

$$f(x) = M\, cos\beta\, x + Nsin\beta x \qquad (7')$$

where M and N - are constant numbers.

a) if βi it is not a root of the characteristic equation, then a particular solution should be sought in the form

$$y^* = Acos\beta x + Bsin\beta x \qquad (8')$$

b) if βi is the root of the characteristic equation, then a particular solution should be sought in the form

$$y^* = x\,(A\cos\beta x + B\sin\beta x) \qquad (9')$$

Note that function $(7')$ is a special case of function (7) $(P(x) = M, Q(x) = N,\ \alpha = 0)$; functions $(8')$ and $(9')$ are special cases of (8) and (9).

Example 4. Find the general integral of a linear inhomogeneous equation

$$y'' + 2y' + 5y = 2\cos x$$

Solution. Characteristic equation

$$k^2 + 2k + 5 = 0$$

has roots

$$k_1 = -1 + 2i,\ k_2 = -1 - 2i.$$

Therefore, the general integral of the corresponding homogeneous equation is

$$\bar{y} = e^{-x}(C_1\cos 2x + C_2\sin 2x)$$

We are looking for a special solution to the inhomogeneous equation in the form

$$y^* = A\cos x + B\sin x$$

where A and B – are constant coefficients to be determined.

Substituting y^* into the given equation, we will have

$$-A\cos x - B\sin x + 2(-A\sin x + B\cos x) + 5(A\cos x + B\sin x) = 2\cos x$$

Equating the coefficients cosx and sinx, we obtain two equations for determining A and B:

$$-A + 2B + 5A = 2, \qquad -B - 2A + 5B = 0$$

whence $A = \frac{2}{5}$, $B = \frac{1}{5}$. The general solution to this equation is: $\bar{y} = y + y^*$

that is

$$y = e^{-x}(C_1\cos 2x + C_2\sin 2x) + \frac{2}{5}\cos x + \frac{1}{5}\sin x$$

Example 5. Solve the equation $y'' + 4y = \cos 2x.$

Solution. The characteristic equation has roots $k_1 = 2i$, $k_2 = -2i$; Therefore, the general solution of the homogeneous equation has the form

$$\bar{y} = C_1 cos2x + C_2 sin2x$$

We look for a particular solution of the inhomogeneous equation in the form

$$y^* = x(Acos2x + Bsin2x)$$

Then

$$y^{*\prime} = 2x(-Asin2x + Bcos2x) + (Acos2x + Bsin2x)$$

$$y^{*\prime\prime} = 4x(-Acos2x - Bsin2x) + 4(-Asin2x + Bcos2x)$$

Substituting these expressions of derivatives into this equation and equating the coefficients at $cos2x$ and $sin2x$, we obtain a system of equations for determining $A\ and\ B$; $4B = 1,\ -4A = 0,\ whence\ A = 0,\ B = \frac{1}{4}$.

Thus, the general integral of this equation

$$y = C_1 cos2x + C_2\ sin2x + \frac{1}{4}xsin2x .$$

Example 6. Solve the equation $y'' - y = 3e^{2x}cosx$

Solution. The right side of the equation has the form

$$f(x) = e^{2x}(M\,cosx + N\,sinx)$$

where $M = 3,\ N = 0$. The characteristic equation $k^2 - 1 = 0$ has roots $k_1 = 1$, $k_2 = -1$. The general solution to the homogeneous equation is

$$\bar{y} = C_1 e^{x} + C_2 e^{-x}$$

Since the number $\alpha + i\beta = 2 + i \cdot 1$ is not a root of the characteristic equation, we look for a particular solution in the form

$$y* = e^{2x}(Acosx + Bsinx)$$

Substituting this expression into the equation, we obtain after bringing similar terms

$$(2A + 4B)e^{2x}cosx + (-4A + 2B)e^{2x}sinx = 3e^{2x}cosx.$$

Equating the coefficients cosx and sinx, we get

$$2A + 4B = 3, \quad -4A + 2B = 0$$

Hence $A = \frac{3}{10}$, $B = \frac{3}{5}$. Therefore, the particular solution

$$y* = e^{2x}\left(\frac{3}{10}cosx + \frac{3}{5}sinx\right)$$

in general

$$y = C_1 e^x + C_2 e^{-x} + e^{2x}\left(\frac{3}{10}cosx + \frac{3}{5}sinx\right)$$

Try to decide for yourself [3]

1. Find a partial solution of the equation $y'' - 2y' - 3y = e^{4x}$, satisfying the boundary conditions $y|_{x=ln2} = 1$; $y|_{x=2ln2} = 1$.
2. Integrate the equation $y'' + y' - 2y = cosx - 3sinx$ under the initial conditions $y(0) = 1$, $y'(0) = 2$.
3. Integrate the equation $y'' - y' = ch2x$ under the initial conditions $y(0) = y'(0) = 0$.

Answers.

1) $y = \frac{1}{5}e^{4x} + \frac{652}{75}e^{-x} - \frac{491}{600}e^{3x}$

2) $y = e^x + sinx$

3) $y = -\frac{1}{3}e^x + \frac{1}{3}ch2x + \frac{1}{6}sh2x$

8 - §. Inhomogeneous linear equations of higher orders

Consider the equation

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = f(x) \qquad (1)$$

where $a_1, a_2, \ldots, a_n, f(x)$ − are continuous functions of from x. Let us know the general solution

$$\bar{y} = C_1y_1 + C_2y_2 + \cdots + C_ny_n \qquad (2)$$

corresponding homogeneous equation [1]

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = 0 \qquad (3)$$

Theorem. If is the general solution of homogeneous equation (3), and $y*$ − is a particular solution of inhomogeneous equation (1), then

$$y = \bar{y} + y^*$$

is a general solution to an inhomogeneous equation.

Thus, the problem of integrating equation (1), as in the case of a second-order equation, is reduced to finding a particular solution to the inhomogeneous equation.

As in the case of a second-order equation, a particular solution to equation (1) can be found by varying arbitrary constants, considering in expression (2) $C_1, C_2, \ldots, C_n$ as functions of from x.

Let's create a system of equations

$$\begin{cases} C_1'y_1 + C_2'y_2 + \cdots + C_n'y_n = 0 \\ C_1'y_1' + C_2'y_2' + \cdots + C_n'y_n' = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_1'y_1^{(n-2)} + C_2'y_2^{(n-2)} + \cdots + C_n'y_n^{(n-2)} = 0 \\ C_1'y_1^{(n-1)} + C_2'y_2^{(n-1)} + \cdots + C_n'y_n^{(n-1)} = f(x) \end{cases} \qquad (4)$$

This system of equations with unknown functions $C_1', C_2', \ldots, C_n'$ has well-defined solutions.

So, system (4) can be solved with respect to functions $C_1', C_2', \ldots, C_n'$.

Finding and integrating, we get

$$C_1 = \int C_1'dx + \bar{C}_1 \qquad C_2 = \int C_2'dx + \bar{C}_2, \ldots, C_n = \int C_n'dx + \bar{C}_n$$

where $\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_n$ − are integration constants.

Let us prove that in this case the expression

$$y^* = C_1y_1 + C_2y_2 + \cdots + C_ny_n \qquad (5)$$

is a general solution to the inhomogeneous equation (1).

We differentiate expression (5) n times, taking into account equalities (4) each time; then we will have

$$y^* = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

$$y^{*\prime} = C_1 y'_1 + C_2 y'_2 + \cdots + C_n y'_n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y^{*(n)} = y^{*(n)} = C_1 y_1^{(n)} + C_2 y_2^{(n)} + \cdots + C_n y_n^{(n)} + f(x)$$

Multiplying the terms of the first, second, $C_1, C_2, \ldots, C_n$ and finally, the last equation by $a_n, a_{n-1}, \ldots, a_1$ and 1, respectively, and adding, we get

$$y^{*(n)} + a_1 y_1^{*(n-1)} + a_2 y_2^{*(n-2)} + \cdots + a_n y_n^* = f(x)$$

Since $y_1, y_2, \ldots, y_n$ − are partial solutions of a homogeneous equation, and therefore the sums of terms obtained by adding along the vertical columns are equal to zero.

Therefore, the function

$$y^* = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

where $C_1, C_2, \ldots, C_n$ −are the functions of otx defined by equations (4) is a solution to the inhomogeneous equation (1). This solution depends on n arbitrary constants $\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_n$. As in the case of a second-order equation, it is proved that this is a general solution.

Example 1. Find a general solution to the equation $y^{IV} - y = x^3 + 1$.
Solution. The characteristic equation $k^4 - 1 = 0$ has roots

$$k_1 = 1, \quad k_2 = -1, \quad k_3 = i, \quad k_4 = -i$$

we find a general solution to the homogeneous equation

$$\bar{y} = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$$

We look for a particular solution of the inhomogeneous equation in the form

$$y^* = A_0x^3 + A_1x^2 + A_2x + A_3$$

Differentiating $y*$ four times and substituting the resulting expressions into the given equation, we get

$$-A_0x^3 - A_1x^2 - A_2x - A_3 = x^3 + 1$$

Let us equate the coefficients at the same degrees *x*:

$$-A_0 = 1, \qquad -A_1 = 0, \qquad -A_2 = 0, \qquad -A_3 = 1$$

Hence,

$$y^* = -x^3 - 1$$

the general integral of the inhomogeneous equation is found by the formula

$$y = \bar{y} + y*$$

that is

$$y = C_1e^x + C_2e^{-x} + C_3cosx + C_4sinx - x^3 - 1$$

Example 2. Solve equation $y^{IV} - y = 5cosx$

Solution. The characteristic equation $k^4 - 1 = 0$ has roots $k_1 = 1,\ k_2 = -1,$ $k_3 = i\ ,\ k_4 = -i.$ Therefore, the general solution to the corresponding homogeneous equation is:

$$\bar{y} = C_1e^x + C_2e^{-x} + C_3cosx + C_4sinx$$

Further, the right-hand side of this inhomogeneous equation has the form

$$f(x) = Mcos\,x + Nsinx$$

where $M = 5,\ N = 0.$

Since i is a simple root of the characteristic equation, we look for a particular solution in the form

$$y* = x\,(Acosx + Bsinx)$$

Substituting this expression into the equation, we find

$$4Asinx - 4Bcosx = 5\,cosx$$

where

$$4A = 0, \qquad -4B = 5 \ \ or\ \ A = 0, B = -\frac{5}{4}.$$

Therefore, a particular solution to the differential equation is

$$y* = -\frac{5}{4}x\,sinx$$

but by a general decision

$$y = C_1e^x + C_2e^{-x} + C_3cosx + C_4sinx - \frac{5}{4}xsinx$$

Try to decide for yourself [3]

1. Solve the equation $y'' - 2y' + 2y = x^2$.

2. Solve the equation $y'' + y = xe^x + 2e^{-x}$

3. Solve the equation $y''' + y'' - 2y' = x - e^x$

4. Find a solution to the equation $y'' + y = 3\,sinx$, satisfying the boundary conditions $y(0) + y'(0) = 0$, $y(\frac{\pi}{2}) + y'(\frac{\pi}{2}) = 0$.

Answers.

1) $y = e^x(C_1\ cosx + C_2\ sinx) + \frac{1}{2}(x+1)^2$

2) $y = C_1\ cosx + C_2\ sinx + \frac{1}{2}(x-1)e^x + e^{-x}$

3) $y = C_1 + C_2\ e^x + C_3\ e^{-2x} - \frac{1}{4}x(x+1) - \frac{1}{3}xe^x$

4) $y = \frac{1}{2}[(\pi+2)cosx - (\pi-2)sinx] - \frac{3}{2}xcosx$

9 - §. Euler's differential equation with variable coefficients

These are differential equations

$$x^2y'' + pxy' + qy = 0 \qquad (1)$$

where p and q are constants. Equations (1) are called Euler equations. [7].

The solutions to differential equation (1) are similar to those used for differential equations with constant coefficients. For these latter, the solution had to be similar to its derivatives. For differential equation (1), the derivatives y' and y'' should become similar after multiplying y them with x and x^2, respectively.

A function has this property. Substituting its left side (1), we find $y = x^k$

$$x^k[k(k-1) + pk + q].$$

For this expression to be identical to zero, k must be the root of the quadratic equation

$$k(k-1) + pk + q = 0 \qquad (2)$$

which is an analogue of the characteristic equation. If (2) has different real roots $k_1 = a,\ k_2 = b$, the solution to (2) is:

$$y = C_1x^a + C_2x^b \qquad (3)$$

If the roots (2) have the form

$$k_{1,2} = a \pm bi$$

instead of (3) we have

$$y = C'_1x^{a+bi} + C'_2x^{a-bi} = x^a(C'_1x^{bi} + C'_2x^{-bi}) \quad (4)$$

Where C'_1 and C'_2 are arbitrary constants. Since then $x = e^{lnx}$,

$$x^{bi} = e^{(blnx)i} = \cos(blnx) + isin(blnx)$$

$$x^{-bi} = e^{-(blnx)i} = \cos(blnx) - isin(blnx)$$

Substituting in (4) the names of constants, we find

$$y = x^a[C_1\cos(blnx) + C_2\sin(blnx)]$$

Finally, if (2) has only one root $k = a,$ then we must consider equation (1) as a limit for $\Delta a \to 0$ the equation with solutions

$$y_1 = x^a,\ y_2 = x^{a+\Delta a}$$

The solution to the last equation will be the function

$$\frac{x^{a+\Delta a} - x^a}{\Delta a}$$

The solution to equation (1) in the case of interest to us will be

$$\lim_{\Delta a \to 0} \frac{x^{a+\Delta a} - x^a}{\Delta a} = \frac{\partial (x^a)}{\partial a} = x^a ln x$$

Hence the general solution (1) will be

$$y = C_1 x^a + C_2 x^a ln x$$

Examples.

1. $x^2 y'' - 8xy' + 20y = 0$. Here equation (2) looks like

$$k(k-1) - 8k + 20 = 0,$$

$$k^2 - 9k + 20 = 0$$

$$D = (-9)^2 - 4 \cdot 1 \cdot 20 = 81 - 80 = 1$$

$$k_1 = \frac{9-1}{2 \cdot 1} = 4, \qquad k_2 = \frac{9+1}{2 \cdot 1} = 5$$

and the general solution of the differential equation

$$y = C_1 x^4 + C_2 x^5 \qquad \blacksquare$$

2. $x^2 y'' - 3xy' + 4y = 0$. Here equation (2) looks like

$$k^2 - 4k + 4 = 0$$

single root $k = 2$. Hence, the general solution of the differential equation

$$y = C_1 x^2 + C_2 x^2 ln x$$

3. $x^2 y'' - xy' + 2y = 0$.Equation (2) will be.Its roots are equal. This means that the general solution of the differential equation $k^2 - 2k + 2 = 0$;
$k_{1,2} = 1 \pm i$

$$y = x(C_1 cos ln x + C_2 sin ln x). \ \blacksquare$$

10 - §. Approximate solution of differential equation Euler–Cauchy method

$$y' = f\ (x, y) \qquad (1)$$

followed by an initial condition [13]

$$y|_{x=x_0} = y_0 \qquad (2)$$

Since the value y_0 of the said solution corresponds to the value x_0, then the matter comes down to finding the difference $y_1 - y_0$, that is Δy, the increment caused by the increment $\Delta x = x_1 - x_0$. But it's Δx very little. Then, Δy with great accuracy, we can imagine the calculation of the auto point x_0, based on which the argument x received an increment. The function is a solution to equation (1), then for each we have $y' = f(x, y)$, where y, represents the value of the function that corresponds exactly to this. For $x = x_0$ will be

$$y' = f(x_0, y_0)$$

and that's why

$$\Delta y \cong f(x_0, y_0)\Delta x$$

We find the replacement Δy *and* $\Delta x y_1 - y_0$ *and* $x_1 - x_0$

$$y_1 \cong y_0 + f(x_0, y_0)(x_1 - x_0) \qquad (3)$$

This is the basic calculation formula using the Euler–Cauchy method. Its accuracy is higher, the smaller the difference $x_1 - x_0$.

Using formula (3) we passed from the value of y_0 to the value of y_1, we can go from the value of y_1 to the value of y_2 of our solution, which corresponds to argument x_2, close to much x_1.

We illustrate the above with three examples.

1) Let $y = y(x)$ be the solution of the differential equation

$$y' = 2\frac{y}{x} \qquad (4)$$

which satisfies the condition

$$y|_{x=1} = 1 \qquad (5)$$

I'll find y (2).

Since the interval k from $x = 1$ before $x = 2$ cannot be considered small, we divide it into 10 equal parts by points $x_1 = 1{,}2$, $x_2 = 1{,}2$, ...

By formula (3)

$$y_1 = 1 + 2 \cdot \frac{1}{1} \cdot 0{,}1 = 1{,}2$$

The remaining values of the function $y(x)$ are found similarly, and it is convenient to place the calculations in the following table, the structure of which will not require any further Δy *accepted* $y'\Delta x$ explanation

X	*at*	*y′*	Δy
1	1	2	0,2
1,1	1,2	2,18	0,218
1,2	1,418	2,36	0,236
1,3	1,654	2,54	0,254
1,4	1,908	2,73	0,273
1,5	2,181	2,91	0,291
1,6	2,472	3,09	0,309
1,7	2,781	3,27	0,327
1,8	3,108	3,45	0,345
1,9	3,453	3,63	0,363
2	3,816		

This table shows that

$$y(2) = 3{,}816$$

Equation (4) is easily solved by separating variables, which gives

$$\frac{dy}{y} = \frac{2dx}{x}$$

where

$$lny = 2lnx + lnC, \qquad thet\ \ is\ y = Cx^2$$

To satisfy condition (5), we must take $C = 1$. Thus, a particular solution is $y = x^2$. The exact value of $y(2) = 4$. The absolute error of the value $y(2) = 3{,}816$ obtained by the method Euler–Cauchy, equal to 0,184, arelative

$$\delta = \frac{0{,}184}{4} = 0{,}046 = 4{,}6\%$$

2) Apply the method to find $y(2)$, if y(x) is a solution to the differential equation

$$y' = \frac{y}{x} + \frac{x}{10} \qquad (6)$$

satisfying the condition

$$y|_{x=1} = 0{,}1 \qquad (7)$$

Divide the segment from $x = 1$ before $x = 2$ into 10 equal parts and make a table.

x	*y*	*y′*	Δy
1	0,1	0,2	0,02
1,1	0,12	0,219	0,022

1,2	0,142	0,238	0,024
1,3	0,166	0,258	0,026
1,4	0,192	0,277	0,028
1,5	0,220	0,297	0,030
1,6	0,250	0,316	0,032
1,7	0,282	0,336	0,034
1,8	0,316	0,356	0,036
1,9	0,352	0,375	0,038
2	0,39	-	-

It follows from the table that $y(2) = 0{,}39$. On the other hand, solving differential equation (6) by substitution, we have $y = uv$

$$u'v + u\left(v' - \frac{v}{x}\right) = \frac{x}{10}$$

From here we first find $v = x,\ and\ then\ u = \frac{x}{10} + C$. This means that the general solution to the differential equation is

$$y = \frac{x^2}{10} + Cx$$

and condition (7) gives $C = 0$. Particular solution

$$y = \frac{x^2}{10}$$

But then $y(2) = 0{,}4$. The absolute error of the value $y(2) = 0{,}39$, found by the Euler–Cauchy method, is equal to $0{,}01$, and the relative error

$$\delta = \frac{0{,}01}{0{,}4} = 0{,}025 = 2{,}5\%$$

3) In the examples considered, exact solutions of differential equations could be found. Let us now consider the Riccati equation

$$y' = x - y^2$$

which is not even integrated in quadratures. Let $y = y(x)$ be that particular solution for which $y(1) = 1$. Let us find (1,5) using the method

Euler - Cauchy, dividing the segment $[1, \frac{3}{2}]$ by points 1, $1\frac{1}{10}, 1\frac{2}{10}, 1\frac{3}{10}, 1\frac{4}{10}, 1\frac{5}{10}$ into 5 parts. The results are visible from the table.

X	*at*	y'	Δy
1	1	0	0
1,1	1	0,1	0,01
1,2	1,01	0,18	0,018
1,3	1,028	0,24	0,024
1,4	1,052	0,30	0,030
1,5	1,082		

So, $y(1,5) = 1,082$. Actually $y(1,5) = 1,091 \ldots$. The absolute error of the found value of the equation is 0,009, and the relative

$$\delta = \frac{0,009}{1,091} < 0,009 = 0,9\%$$

The error in formula (3) occurs from the substitution. If Δy *on* $y'\Delta x$ $y(x_0 + \Delta x)$ apply the Taylor formula with the remainder term, we get

$$y(x_0 + \Delta x) = y(x_0) + y'(x_0)\Delta x + \frac{1}{2} y''(\bar{x})(\Delta x)^2$$

where $\bar{x}$ lies between x_0 *and* $x_0 + \Delta x$. Hence, assuming

$$y(x_0 + \Delta x) - y(x_0) = \Delta y$$

we find

$$\Delta y - y'(x_0)\Delta x = \frac{1}{2} y''(\bar{x})(\Delta x)^2$$

This difference is of the order of magnitude $(\Delta x)^2$. The error of formula (3) decreases Δx and *n* decreases approximately n^2 once. The total error will decrease n times. An increase in the number of intermediate points entails a decrease in the total error. If then $n \to \infty$, the total error tends to zero

Try to decide for yourself [3]

1. Using the Euler–Koshin method $y|_{x=0,4}$, find if $y = \frac{2xy}{x^2+1}$ *and* $y|_{x=0} = 1$. Segment [0; 0,4] split into 4 equal parts.

Answer.1,12

2. Find the relative error of solving the previous problem.
   Answer: 3,4%

3. Using the Euler–Koshin method $y|_{x=1,4}$, find if $y' = \frac{2y}{x} + x^2$, $y|_{x=1} = 1$. Segment [1; 1,4] split into 4 equal parts.

   Answer: 2,57

4. Find the relative error of solving the previous problem.
   Answer: 6,2%

5. Using the Euler–Koshin method $y|_{x=2,4}$, find if $y' = \frac{2x+y^2-5}{2}$, $y|_{x=2} = 1$. Segment [2; 2,4] split into 4 equal parts.

   Answer: 1,06

6. Solve the same problem by breaking [2; 2,4] into 8 equal parts.

   Answer: 1,077

11 - §. Euler method

Let a differential equation be given [2]

$$y' = f(x, y) \quad (1)$$

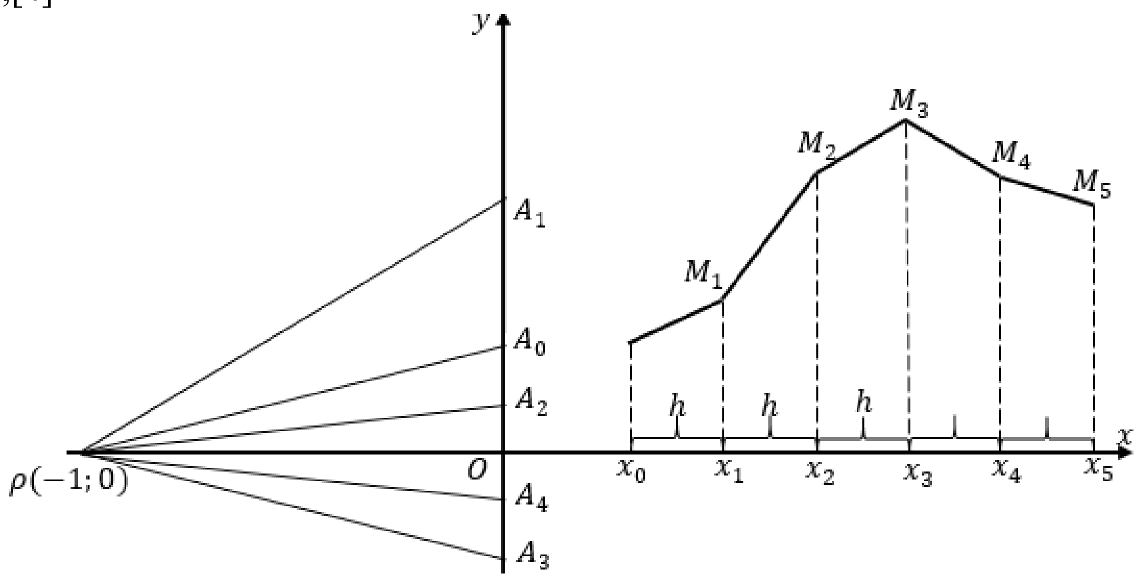
with the initial condition

$$y(x_0) = y_0$$

By choosing a small enough steph, let's build a system of equally spaced points

$$x_i = x_0 + ih \quad (i = 0,1,2, \dots.) \qquad (2)$$

The required integral curve $y = y(x)$, passing through the point $M_0(x_0, y_0)$, let us approximately replace (Fig. 1) the broken line $M_0M_1M_2$......, [3],[4]



the tops of the $M_i(x_i, y_i)$ $(i = 0,1,2, \dots)$, links which are M_iM_{i+1} straight between straight lines $x = x_i$, $x = x_{i+1}$ and have a rise

$$\frac{y_{i+1} - y_i}{h} = f(x_i, y_i) \qquad (3)$$

(the so-called Euler's polygon).

The links M_iM_{i+1} of the Euler manifold at each M_i vertex have a direction coinciding with the direction $y_i' = f(x_i, y_i)$ of the integral curve of equation (1) passing through the point M_i.

From formula (3) it follows that the y_i values can be determined (Euler's method) using the formulas

$$y_{i+1} = y_i + \Delta y_i$$

and

$$\Delta y_i = hf(x_i, y_i) \quad (i = 0,1,2, \dots) \qquad (4)$$

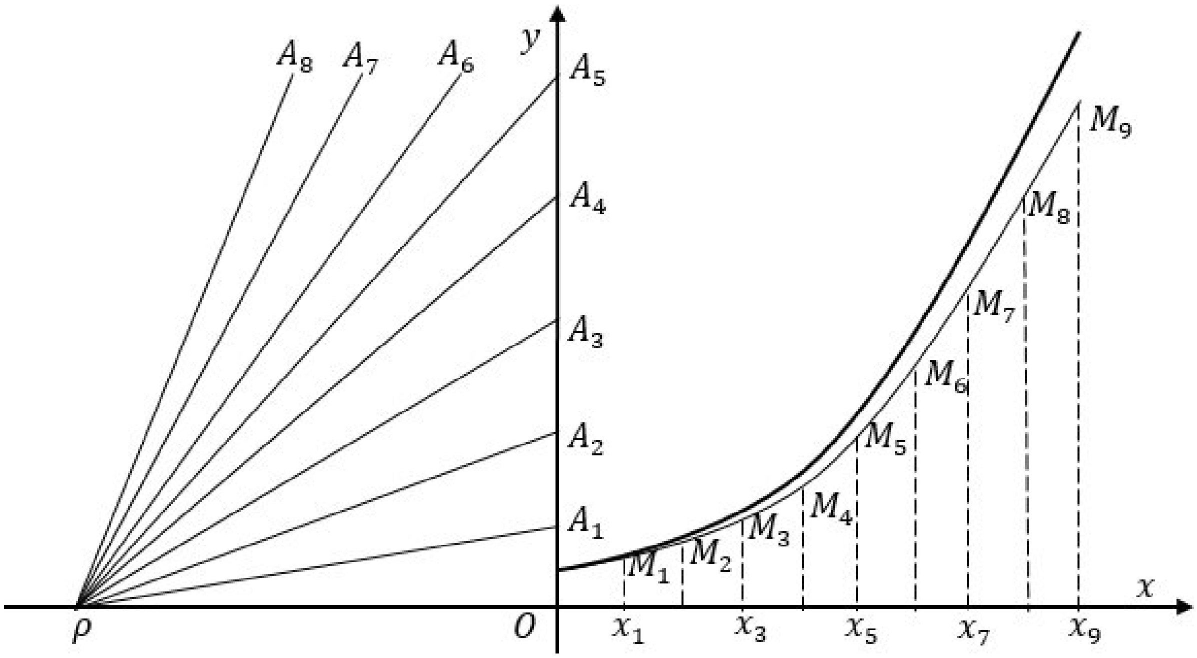
For the geometric construction of the Euler polygon, we choose the pole $P(-1, 0)$ and on the ordinate axis we plot the segment $OA_0 = f(x_0, y_0)$ (Fig. 2).

The angular coefficient of the beam PA_0 will be equal to $f(x_0, y_0)$; to obtain the first link of Euler's broken line, it is enough to draw a straight line M_0M_1 from point M_0, parallel to the ray PA_0, until it intersects with the straight line $x =$

x_1 at some point $M_0(x_0,\ y_0)$. Taking point $M_1(x_1,\ y_1)$ as the initial one, we lay off the segment $OA_1 = f(x_1,\ y_1)$ on the ordinate axis and through point M_1 we draw a line $M_1M_2 || OA_1$ until it intersects at point M_2 straight line $x = x_2$ etc.

Euler's method is the simplest numerical method for integrating a differential equation. Disadvantages:

1) low accuracy;
2) systematic accumulation of errors.



If the right-hand side $f(x,\ y)$ of equation (1) is continuous, then the sequence of Euler broken $h \to 0$ lines for $[x_0,\ x_0 + H]$ a sufficiently small intervaluniformly tends to the pussy integral curve $y = y(x)$.

Example. Using Euler's method, compile a table of values of the integral of the differential equation on the interval [0; 1]

$$y' = \frac{xy}{2} \qquad (5)$$

satisfying the initial condition $y(0) = 1$, choosing step $h = 0{,}1$.

Solution. The calculation results are presented in the table. For comparison, the last column of the chain contains the values of the exact solution

$$y = e^{\frac{1}{4}x^2} \qquad (6)$$

Integrating differential equation (5) using the Euler method

i	x	y	$f(x,y) = \frac{xy}{2}$	$\Delta y = 0{,}1f(x,y)$	Exact value

					$y = e^{\frac{x^2}{4}}$
0	0	1	0	0	1
1	0,1	1	0,05	0,005	1,0025
2	0,2	1,005	0,1005	0,0101	1,0100
3	0,3	1,0151	0,1523	0,0152	1,0227
4	0,4	1,0303	0,2067	0,0206	1,0408
5	0,5	1,0509	0,2627	0,0263	1,0645
6	0,6	1,0772	0,3232	0,0323	1,0942
7	0,7	1,1095	0,3883	0,0388	1,1303
8	0,8	1,1483	0,4593	0,0459	1,1735
9	0,9	1,1942	0,5374	0,0537	1,2244
10	1,0	1,2479			1,2840

From the table above it is clear that the absolute error of the value y_{10} is ε_{10}= 0,0361. Hence the relative error is approximately 3%.

For comparison, we present a graph of the exact solution (highlighted with a thick line) and the corresponding Euler polyline $M_0M_1M_2$... (Fig. 2).

Euler's method has low accuracy and produces relatively satisfactory results only for small values of h. Essentially, Euler's method consists in the fact that the integral of the differential equation (1) on each partial interval $[x_i, x_{i+1}]$ is represented by two terms of the Taylor series

$$y(x_i + h) = y(x_i) + hy'(x_i) \quad (i = 0,1,2,\dots)$$

that is, for this segment there is an error of the order of h^2.

When calculating the values in the next segment, the original data are not accurate and contain errors that depend on the inaccuracy of the previous calculations.

Examples for self-solution

1. Find an approximate solution of the equation $y' = y + x$ on the segment $[0, 1]$, satisfying the initial conditions $x_0 = 0$, $y_0 = 1$ and calculate y at $x = 1$.

2. Using the Euler method, compile a table of approximate eigenvalues of this equation $y' = 0{,}5xy$, satisfying the initial condition $y\ (0) = 1$ with step $h = 0{,}1$ on the interval [0, 1].

3. Using the Euler method, find three values of the function y, defined by the equation $y' = 1 + x + y^2$, under the initial condition $y\ (0) = 1$, assuming $h = 0{,}1$.

4. Using the Euler method, find four values of the function y, defined by the equation $y' = x^2 + y^2$, with the initial condition $y(0) = 0$, assuming $h = 0{,}1$.

5. Using the Euler method, find a numerical solution to the equation $y' = y^2 + \frac{y}{x}$ with the initial condition $y\ (2) = 4$, assuming $h = 0{,}1$ (four values).

Answers. 1)

x	0	0,1	0,2	0,3	0,4
y	1	1,1	1,22	1,36	1,52

2) $y = 1{,}2840 = e^{\frac{x^2}{4}}$, fromthis $y(1) = e^{\frac{1}{4}}$

3)

x	0,1	0,2	0,3	0,4
y	1	1,2	1,45	1,78

4)

x	0,1	0,2	0,3	0,4
y	0	0,001	0,005	0,014

5

5)

x	2	2,1	2,2	2,3	2,4
y	4	5,8	9,44	18,78	54,86

13- §. Systems of ordinary differential equations

When solving many problems, it is necessary to find the functions

$y_1 = y_1(x),\ y_2 = y_2(x),\ \dots,\ y_n = y_n(x)$, which satisfy a system of differential equations containing arguments, the desired functions $y_1, y_2, \dots, y_n$ and their derivatives. [1]

Consider the system of first-order equations

$$\begin{cases} \frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n) \\ \frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots, y_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{dy_n}{dx} = f_n(x, y_1, y_2, \dots, y_n) \end{cases} \qquad (1)$$

where $y_1, y_2, \dots, y_n$ – are the required functions, x- is the argument.

Such a system, when the left side of the equations contains first-order derivatives, and the right sides do not contain derivatives, is called normal.

To integrate a system means to determine functions $y_1, y_2, \dots, y_n$ that satisfy the system of equations (1) given initial conditions

$$y_1|_{x=x_0} = y_{10},\ y_2|_{x=x_0} = y_{20}, \dots,\ y_n|_{x=x_0} = y_{n0} \qquad (2)$$

Integration of a system of type (1) can be done as follows.
Let us differentiate with respect to the first equation (1):

$$\frac{d^2y_1}{dx^2} = \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y_1}\frac{dy_1}{dx} + \dots + \frac{\partial f_1}{\partial y_n}\frac{dy_n}{dx}$$

Replacing their derivatives $\frac{dy_1}{dx}, \frac{dy_2}{dx}, \dots, \frac{dy_n}{dx}$ with expressions $f_1, f_2, \dots, f_n$ from equations (1), we will have the equation

$$\frac{d^2y_1}{dx^2} = F_2(x, y_1, \dots, y_n)$$

Differentiating the resulting equation and proceeding similarly to the previous one, we find

$$\frac{d^3y_1}{dx^3} = F_3(x, y_1, \dots, y_n)$$

Continuing further, in the same way we finally obtain the equation

$$\frac{d^n y_1}{dx^n} = F_n(x, y_1, \dots, y_n)$$

So we get the following system:

$$\begin{cases} \frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n) \\ \frac{d^2 y_1}{dx^2} = F_2(x, y_1, y_2, \dots, y_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{d^n y_1}{dx^n} = F_n(x, y_1, y_2, \dots, y_n) \end{cases} \quad (3)$$

From the first $n-1$ equations we define $y_2, y_3, \dots, y_n$, express them through x, y_1 and derivatives $\frac{dy_1}{dx}, \frac{d^2 y_1}{dx^2}, \dots, \frac{d^{n-1} y_1}{dx^{n-1}}$ (it is assumed that these operations are feasible):

$$y_2 = \varphi_2(x, y_1, y_1', \dots, y_1^{(n-1)})$$

$$y_3 = \varphi_3(x, y_1, y_1', \dots, y_1^{(n-1)})$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_n = \varphi_n(x, y_1, y_1', \dots, y_1^{(n-1)})$$

Substituting these expressions into the last of equations (3), we obtain an n th order equation for determining y_1

$$\frac{d^n y_1}{dx^n} = \Phi(x, y_1, y_1', \dots, y_1^{(n-1)}) \quad (5)$$

Solving this equation, we determine y_1:

$$y_1 = \psi_1(x, C_1, C_2, \dots, C_n)$$

Differentiating the last expression $n-1$ times, we find $\frac{dy_1}{dx}, \frac{d^2 y_1}{dx^2}, \dots, \frac{d^{n-1} y_1}{dx^{n-1}}$ the derivativesas functions from $x, C_1, C_2, \dots, C_n$. Substituting these functions into equation (4), defined by $y_2, y_3, \dots, y_n$:

$$\begin{aligned} y_2 &= \psi_2(x, C_1, C_2, \dots, C_n) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n &= \psi_n(x, C_1, C_2, \dots, C_n) \end{aligned} \quad (7)$$

In order for the resulting solution to satisfy the given initial conditions (2), all that remains is to find from equations (6) and (7) the corresponding values of the constants $C_1, C_2, \dots, C_n$.

Example 1. Integrate the system

$$\frac{dy}{dx} = y + z + x, \quad \frac{dz}{dx} = -4y - 3z + 2x \qquad \text{(a)}$$

initial conditions

$$y|_{x=0} = 1, \quad z|_{x=0} = 0 \qquad \text{(b)}$$

Solution. 1) Differentiating with respect to the first equation, we will have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{dz}{dx} + 1$$

Substituting expressions from $\frac{d}{dx}$ and $\frac{d}{dx}$ equations (a) here, we get

$$\frac{d^2y}{dx^2} = (y + z + x) + (-4y - 3z + 2x) + 1$$

or

$$\frac{d^2y}{dx^2} = -3y - 2z + 3x + 1 \qquad \text{(c)}$$

2) From the first equation of the system (a) we find

$$z = \frac{dy}{dx} - y - x \qquad \text{(d)}$$

and substitute into the equation we just obtained; we get

$$\frac{d^2y}{dx^2} = -3y - 2\left(\frac{dy}{dx} - y - x\right) + 3x + 1$$

or

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 5x + 1 \qquad \text{(e)}$$

The general solution to the last equation is

$$y = (C_1 + C_2x)e^{-x} + 5x - 9 \qquad \text{(f)}$$

on the basis (d)

$$z = (C_2 - 2C_1 - 2C_2x)e^{-x} - 6x + 14 \qquad \text{(g)}$$

Let us select the constants C_1 and C_2 so that the initial conditions (b) are satisfied: $y|_{x=0} = 1$, $z|_{x=0} = 0$. Then from equalities (f) and (g) we obtain

$$1 = C_1 - 9, \qquad 0 = C_2 - 2C_1 + 14$$

whence $C_1 = 10$, $C_2 = 6$. Thus, the solution satisfying the given initial conditions (b) has the form

$$y = (10 + 6x)e^{-x} + 5x - 9, \quad z = (-14 - 12x)e^{-x} - 6x + 14$$

Example 2. Integrate the system

$$\frac{dx}{dt} = y + z, \qquad \frac{dy}{dt} = x + z, \qquad \frac{dz}{dt} = x + y$$

Solution. Differentiating by *t* the first equation, we find

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} + \frac{dz}{dt} = (x + z) + (x + y), \qquad \frac{d^2x}{dt^2} = 2x + y + z$$

Excluding variables y and z from equations

$$\frac{dx}{dt} = y + z, \qquad \frac{d^2x}{dt^2} = 2x + y + z$$

we will have a second-order equation with respect to x

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0$$

Integrating this equation, we obtain its general solution

$$x = C_1e^{-t} + C_2e^{2t} \quad (\alpha)$$

from here we find

$$\frac{dx}{dt} = -C_1e^{-t} + 2C_2e^{2t} \quad \text{and} \quad y = \frac{dx}{dt} - z = -C_1e^{-t} + 2C_2e^{2t} \qquad (\beta)$$

Substituting the found expressions for x and y into the third of the given equations, we obtain the equation for determining z

$$\frac{dz}{dt} + z = 3C_2 e^{2t}$$

Integrating this equation, we find

$$z = C_1e^{-t} + C_2e^{2t} \qquad (\gamma)$$

but then based on the equations we get (β)

$$y = -(C_1 + C_2)e^{-t} + C_2e^{2t} \qquad (\delta)$$

Equations $(\alpha), (\beta)$ *and* (γ) give a general solution to a given system.

The differential equations of the system may include derivatives of higher orders. In this case, a system of higher order differential equations is obtained.

So, for example, the problem of the movement of a material point under the influence of force F is reduced to a system of three-differential equations of the second order. Let F_x, F_y, F_z be the projections of force F onto the coordinate axis. The position of a point at any time *t* is determined by its *x, y, z* coordinates. Therefore, x, y, z are functions of *t*. The projections of the velocity vector of the point on the coordinate axis will be $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$.

Let us assume that the force F, and therefore its projections Fx, Fy, Fz, depend on the time*t*, positions x, y, z of the point and on the speed of movement of the point, that is, on $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$.

The functions sought in this problem are three functions

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

These functions are determined from the equations of dynamics (Newton's law)

$$m\frac{d^2x}{dt^2} = F_x(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$$

$$m\frac{d^2y}{dt^2} = F_y(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) \qquad (8)$$

$$m\frac{d^2z}{dt^2} = F_z(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$$

We obtained a system of three - differential equations of the second order. In the case of plane motion, that is, motion when the trajectory is a plane curve (lying, for example, in the Oxy plane), we obtain a system of two equations for determining the functions $x(t)$ and $y(t)$:

$$m\frac{d^2x}{dt^2} = F_x(t, x, y, \frac{dx}{dt}, \frac{dy}{dt}) \qquad (9)$$

$$m\frac{d^2y}{dt^2} = F_y(t, x, y, \frac{dx}{dt}, \frac{dy}{dt}) \qquad (10)$$

It is possible to solve a system of higher - order differential equations by reducing it to a system of first - order equations. Using equations (9) and (10) as an example, we will show how this is done. Let us introduce the notation

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v$$

Then

$$\frac{d^2x}{dt^2} = \frac{du}{dt}, \quad \frac{d^2y}{dt^2} = \frac{dv}{dt}$$

The system of two second-order equations (9), (10) with two desired functions $x\ (t)$ and $y(t)$ is replaced by a system of four first-order equations with four desired functions x, y, u, v

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v$$

$$m\frac{du}{dt} = F_x(t, x, y, u, v), \quad m\frac{dv}{dt} = F_y(t, x, y, u, v).$$

Let us note in conclusion that the general method of solving the system that we have considered can, in some specific cases, be replaced by one or another artificial example that will more quickly lead to the goal.

Example 3. Find a general solution to a system of differential equations

$$\frac{d^2y}{dx^2} = z, \quad \frac{d^2z}{dx^2} = y.$$

Solution. Let us differentiate by two times both sides of the first equation:

$$\frac{d^4y}{dx^4} = \frac{d^2z}{dx^2}$$

But $\frac{d^2z}{dx^2} = y$, therefore, we get a fourth - order equation $\frac{d^4y}{dx^4} = y$. Integrating this equation, we obtain its general solution

$$y = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x$$

Finding from here $\frac{d^2z}{dx^2}$ and substituting into the first equation, we find z:

$$z = C_1e^x + C_2e^{-x} - C_3\cos x - C_4\sin x.$$

Try to decide for yourself [3]

1. Solve a system of differential equations

$$\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = x - y$$

initial conditions $x(0) = 2,\ y(0) = 0$.

2. Solve a system of differential equations

$$\frac{dx}{dt} = \frac{x}{2x + 3y}, \quad \frac{dy}{dt} = \frac{y}{2x + 3y}$$

initial conditions $x(0) = 1,\ y(0) = 2$.

3. Solve a system of differential equations

$$\frac{dx}{dt} = 2y, \quad \frac{dy}{dt} = 2z, \quad \frac{dz}{dt} = 2x$$

4. Solve a system of differential equations

$$\frac{dx}{dt} = 2x + y, \quad \frac{dy}{dt} = x + 2y; \quad x(0) = 1, y(0) = 3.$$

Answers.

1) $x = (\frac{\sqrt{2}}{2} + 1)e^{t\sqrt{2}} + (1 - \frac{\sqrt{2}}{2})e^{-t\sqrt{2}},\ y = \frac{\sqrt{2}}{2}e^{t\sqrt{2}} - \frac{\sqrt{2}}{2}e^{-t\sqrt{2}}$

2) $x = \frac{1}{8}t + 1, y = \frac{1}{4}t + 2$

3) $z = \frac{1}{2}\frac{dy}{dx} = C_1e^{2} - \frac{1}{2}e^{-t}[(C_3\sqrt{3} + C_2)\cos t\sqrt{3} - (C_2\sqrt{3} - C_3)\sin t\sqrt{3}]$

4) $x = 2e^{3t} - e^t, y = 2e^{3t} + e^t$

14 - §. Systems of linear differential equations with constant coefficients

Let us have a system of differential equations

$$\begin{cases} \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{cases} \quad (1)$$

where the coefficients a_{ij} are constant. Here t is the argument $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ are the required functions. System (1) is called a system of linear homogeneous differential equations with constant coefficients. [9].

We will look for a particular solution of the system in the following form:

$$x_1 = \alpha_1 e^{kt}, \quad x_2 = \alpha_2 e^{kt}, \ldots, x_n = \alpha_n e^{kt} \quad (2)$$

It is required to define the constants α_1, α_2, ... , α_n and k so that the functions $\alpha_1 e^{kt}$, $\alpha_2 e^{kt}$, ... , $\alpha_n e^{kt}$ satisfied the system of equations (1). Substituting them into system (1), we get

$$k\alpha_1 e^{kt} = (a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n)e^{kt}$$

$$k\alpha_2 e^{kt} = (a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n)e^{kt}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$k\alpha_n e^{kt} = (a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nn}\alpha_n)e^{kt}\text{k}$$

We reduce e^{kt}. By transferring all terms to one side and collecting the coefficients at $\alpha_1, \alpha_2, \ldots, \alpha_n$, we obtain a system of equations.

$$(a_{11} - k)\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n = 0$$

$$a_{21}\alpha_1 + (a_{22} - k)\alpha_2 + \cdots + a_{2n}\alpha_n = 0 \quad (3)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + (a_{nn} - k)\alpha_n = 0$$

Let us choose α_1, α_2, ... , α_n and k such that system (3) is satisfied. This system is a system of linear homogeneous algebraic equations with respect to $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let's compose the determinant of the system (3):

$$\Delta(k) = \begin{vmatrix} a_{11}-k & a_{12} \dots & a_{1n} \\ a_{21} & a_{22}-k \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} \dots & a_{nn}-k \end{vmatrix} \qquad (4)$$

If k is such that the determinantΔis nonzero, then system (3) has only zero solutions $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$, and therefore formulas (2) give only trivial solutions

$$x_1(t) = x_2(t) = \cdots x_n(t) \equiv 0$$

Thus, we will obtain nontrivial solutions (2) only if k for which the determinant (4) becomes zero. We arrive at the nth order equation to determine k:

$$\begin{vmatrix} a_{11}-k & a_{12} \dots & a_{1n} \\ a_{21} & a_{22}-k \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} \dots & a_{nn}-k \end{vmatrix} = 0 \qquad (5)$$

This equation is called the characteristic equation for system (1), its roots are called the roots of the characteristic equation.

Let's consider several cases.

I. The roots of the characteristic equation are real and distinct.

Let us denote by the $k_1, k_2, \dots, k_n$ roots of the characteristic equation. For each root k_i, we write system (3) and determine the coefficients $\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_n^{(i)}$. It can be shown that one of them is arbitrary and can be considered equal to medicine. Thus, we obtain: for the root k_1 solution of system (1)

$$x_1^{(1)} = \alpha_1^{(1)} e^{k_1 t},\ x_2^{(1)} = \alpha_2^{(1)} e^{k_1 t}, \dots, x_n^{(1)} = \alpha_n^{(1)} e^{k_1 t};$$

for root k_2 solution of system (1)

$$x_1^{(2)} = \alpha_1^{(2)} e^{k_2 t},\ x_2^{(2)} = \alpha_2^{(2)} e^{k_2 t}, \dots, x_n^{(2)} = \alpha_n^{(2)} e^{k_2 t};$$

……………………………………………

for the root k_n solution of the system (1)

$$x_1^{(n)} = \alpha_1^{(n)} e^{k_n t},\ x_2^{(n)} = \alpha_2^{(n)} e^{k_n t}, \dots, x_n^{(n)} = \alpha_n^{(n)} e^{k_n t}.$$

By direct substitution into the equations, one can verify that the system of functions

$$\begin{cases} x_1 = C_1\alpha_1^{(1)}e^{k_1t} + C_2\alpha_1^{(2)}e^{k_2t} + \cdots + C_n\alpha_1^{(n)}e^{k_nt} \\ x_2 = C_1\alpha_2^{(1)}e^{k_1t} + C_2\alpha_2^{(2)}e^{k_2t} + \cdots + C_n\alpha_2^{(n)}e^{k_nt} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = C_1\alpha_n^{(1)}e^{k_1t} + C_2\alpha_n^{(2)}e^{k_2t} + \cdots + C_n\alpha_n^{(n)}e^{k_nt} \end{cases} \qquad (6)$$

where $C_1, C_2, \ldots, C_n$ – are arbitrary constants, is also a solution to the system of differential equations (1). This is the general solution of system (1). It is easy to show that it is possible to find values of the constants at which the solution will satisfy the given initial conditions.

Example 1. Find a general solution to the system of equations

$$\frac{dx_1}{dt} = 2x_1 + 2x_2, \quad \frac{dx_2}{dt} = x_1 + 3x_2$$

Solution. Making up a characteristic equation

$$\begin{vmatrix} 2-k & 2 \\ 1 & 3-k \end{vmatrix} = 0$$

or $k^2 - 5k + 4 = 0$. We find its roots $k_1 = 1, k_2 = 4$. We look for a solution to the system in the form

$$x_1^{(1)} = \alpha_1^{(1)}e^t, \quad x_2^{(1)} = \alpha_2^{(1)}e^t$$

$$x_1^{(2)} = \alpha_1^{(2)}e^{4t}, \quad x_2^{(2)} = \alpha_2^{(2)}e^{4t}$$

We compose system (3) for the root $k_1 = 1$ and determine $\alpha_1^{(1)}$ *and* $\alpha_2^{(1)}$;

$$(2-1)1\alpha_1^{(1)} + 2\alpha_2^{(1)} = 0, \quad \alpha_1^{(1)} + (3-1)\alpha_2^{(1)} = 0$$

or

$$\alpha_1^{(1)} + 2\alpha_2^{(1)} = 0, \quad \alpha_1^{(1)} + 2\alpha_2^{(1)} = 0$$

where $\alpha_2^{(1)} = -\frac{1}{2}\alpha_1^{(1)}$. Assuming $\alpha_1^{(1)} = 1$, web get $\alpha_2^{(1)} = -\frac{1}{2}$. Thus, we have obtained the solution of the system

$$x_1^{(1)} = e^t, \quad x_2^{(1)} = -e^t/2$$

We further compose system (3) for the root $k_2 = 4$ and define $\alpha_1^{(2)}$ *and* $\alpha_2^{(2)}$;

$$-2\alpha_1^{(2)} + 2\alpha_2^{(2)} = 0, \quad \alpha_1^{(2)} - \alpha_2^{(2)} = 0$$

Where from $\alpha_1^{(2)} = \alpha_2^{(2)}$ and $\alpha_1^{(2)} = 1,\ \alpha_2^{(2)} = 1$. We obtain the second solution of the system

$$x_1^{(2)} = e^{4t}, \quad x_2^{(2)} = e^{4t}$$

The general solution of the system will be

$$x_1 = C_1\ e^t + C_2\ e^{4t}, \quad x_2 = -\frac{1}{2}C_1 e^{t} + C_2 e^{4t}.$$

II. The roots of the characteristic equation are different, but among them there are complex ones.

Let there be two complex conjugate roots among the roots of the characteristic equation:

$$k_1 = \alpha + i\beta, \qquad k_2 = \alpha - i\beta$$

Solutions will correspond to these roots

$$x_j^{(1)} = \alpha_j^{(1)} e^{(\alpha+i\beta)t} \qquad (j = 1,2,\dots,n) \qquad (7)$$

$$x_j^{(2)} = \alpha_j^{(2)} e^{(\alpha-i\beta)t} \qquad (j = 1,2,\dots,n) \quad (8)$$

The coefficients $\alpha_j^{(1)} and\ \alpha_j^{(2)}$ are determined from the system of equations (3). Thus, we obtain two partial solutions

$$\bar{x}_j^{(1)} = e^{\alpha t}(\lambda_j^{(1)} cos\beta x + \lambda_j^{(2)} sin\beta x)$$

$$\bar{x}_j^{(2)} = e^{\alpha t}(\bar{\lambda}_j^{(1)} cos\beta x + \bar{\lambda}_j^{(2)} sin\beta x) \qquad (9)$$

$where$ $\lambda_j^{(1)}, \lambda_j^{(2)}, \bar{\lambda}_j^{(1)}, \bar{\lambda}_j^{(2)}$ – real numbers defined through. The corresponding combinations of functions (9) will be included in the general solution of the system $\alpha_j^{(1)} and\ \ \alpha_j^{(2)}$.

Example 2 . Find the general solution of the system

$$\frac{dx_1}{dt} = -7x_1 + x_2, \qquad \frac{dx_2}{dt} = -2x_1 - 5x_2$$

Solution. Making up a characteristic equation

$$\begin{vmatrix} -7-k & 1 \\ -2 & -5-k \end{vmatrix} = 0$$

or $k^2 + 12k + 37 = 0$ and find its roots

$$k_1 = -6 + i\,, \qquad k_2 = -6 - i$$

Substituting $k_1 = -6 + i$ into system (3), we find

$$\alpha_1^{(1)} = 1, \qquad \alpha_2^{(1)} = 1 + i$$

We write the solution (7):

$$x_1^{(1)} = e^{(-6+i)t}\,, \quad x_2^{(1)} = (1+i)e^{(-6+i)t} \qquad (7')$$

Substituting $k_2 = 6 - i$ into system (3), we find

$$\alpha_1^{(2)} = 1, \qquad \alpha_2^{(2)} = 1 - i$$

Let's learn the second solution system (8):

$$x_1^{(2)} = e^{(-6-i)t}, \quad x_2^{(2)} = (1-i)e^{(-6-i)t} \qquad (8')$$

Let's rewrite the solution (7′):

$$x_1^{(1)} = e^{-6t}(cost + isint) \quad , x_2^{(1)} = (1+i)e^{-6t}(cost + isint)$$

or

$$x_1^{(1)} = e^{-6t}cost + ie^{-6t}sint,$$

$$x_2^{(1)} = e^{-6t}(cost - sint) + ie^{-6t}(cost + sint),$$

Let's rewrite the solution (8′):

$$x_1^{(2)} = e^{-6t}cost - ie^{-6t}sint,$$

$$x_2^{(2)} = e^{-6t}(cost - sint) - ie^{-6t}(cost + sint)$$

For systems of particular solutions, we can take separate real parts and separate imaginary parts:

$$\bar{x}_1^{(1)} = e^{-6t}cost\ , \quad \bar{x}_2^{(1)} = e^{-6t}(cost - sint)$$

$$\bar{x}_1^{(2)} - e^{-6t}sint, \qquad \bar{x}_2^{(1)} = e^{-6t}(cost + sint) \qquad (9')$$

The general solution of the system will be

$$x_1 = C_1e^{-6t}cost + C_2e^{-6t}sint$$

$$x_2 = C_1e^{-6t}(cost - sint) + C_2e^{-6t}(cost + sint)$$

Using a similar method, you can find a solution to a system of linear differential equations of higher order with constant coefficients.

In the mechanics and theory of electrical circuits, for example, the solution of a system of second-order differential equations is studied

$$\frac{d^2x}{dt^2} = a_{11}x + a_{12}y, \qquad \frac{d^2y}{dt^2} = a_{21}x + a_{22}y \quad (10)$$

Again looking for solutions in form

$$x = \alpha e^{kt}, \quad y = \beta e^{kt}$$

Substituting these expressions into system (10) and reducing to e^{kt}, we obtain a system of equations for determining α, β *and* k

$$(a_{11} - k^2)\alpha + a_{12}\beta = 0, \ a_{21}\alpha + (a_{22} - k^2) = 0 \qquad (11)$$

Non-zero values are α *and* β defined only when the determinant of the system is equal to zero:

$$\begin{vmatrix} a_{11} - k^2 & a_{12} \\ a_{21} & a_{22} - k^2 \end{vmatrix} = 0 \quad (12)$$

This is the characteristic equation for system (10); it is an equation of 4th order relative to k. Let k_1, k_2, k_3 and k_4 - be its roots. For each root k_i from system (11) we find the values α *and* β. The general solution, similar to (6), will have the form

$$x = C_1\alpha^{(1)}e^{k_1 t} + C_2\alpha^{(2)}e^{k_2 t} + C_3\alpha^{(3)}e^{k_3 t} + C_4\alpha^{(4)}e^{k_4 t}$$

$$y = C_1\beta^{(1)}e^{k_1 t} + C_2\beta^{(2)}e^{k_2 t} + C_3\beta^{(3)}e^{k_3 t} + C_4\beta^{(4)}e^{k_4 t}$$

If some of the roots are complex, then each pair of complex roots in the general solution will correspond to expressions of the form (9).

Example 3. Find a general solution to a system of differential equations

$$\frac{d^2x}{dt^2} = x - 4y,$$
$$\frac{d^2y}{dt^2} = -x + y$$

Solution. We write the characteristic equation (12) and find its roots:

$$\begin{vmatrix} 1-k^2 & -4 \\ -1 & 1-k^2 \end{vmatrix} = 0$$

$$k_1 = i, \qquad k_2 = -i, \qquad k_3 = \sqrt{3}, \qquad k_4 = -\sqrt{3}.$$

the solution will be looked for in the form

$$x^{(1)} = \alpha^{(1)}e^{it}, \qquad y^{(1)} = \beta^{(1)}e^{it}$$
$$x^{(2)} = \alpha^{(2)}e^{-it}, \qquad y^{(2)} = \beta^{(2)}e^{-it}$$
$$x^{(3)} = \alpha^{(3)}e^{\sqrt{3}t}, \qquad y^{(3)} = \beta^{(3)}e^{\sqrt{3}t}$$
$$x^{(4)} = \alpha^{(4)}e^{-\sqrt{3}t}, \qquad y^{(4)} = \beta^{(4)}e^{-\sqrt{3}t}$$

From system (11) we find $\alpha^{(j)}$ *and* $\beta^{(j)}$

$$\alpha^{(1)} = 1, \quad \beta^{(1)} = \frac{1}{2}$$
$$\alpha^{(2)} = 1, \quad \beta^{(2)} = \frac{1}{2}$$
$$\alpha^{(3)} = 1, \quad \beta^{(3)} = -\frac{1}{2}$$
$$\alpha^{(4)} = 1, \quad \beta^{(4)} = -\frac{1}{2}$$

Let's write out complex solutions:

$$x^{(1)} = e^{it} = cost + isint, \qquad y^{(1)} = 0{,}5(cost + isint)$$
$$x^{(2)} = e^{-it} = cost - isint, \qquad y^{(2)} = 0{,}5(cost - isint)$$

The solution will be the real imaginary parts:

$$\bar{x}^{(1)} = cost\ ,\quad \bar{y}^{(1)} = 0,5\,cost$$
$$\bar{x}^{(2)} = sint\ ,\quad \bar{y}^{(2)} = 0,5sint$$

Now we can write the general solution

$$x = C_1 \cos t + C_2 \sin t + C_3 e^{\sqrt{3}t} + C_4 e^{-\sqrt{3}t}$$

$$y = \frac{1}{2} C_1 cost + \frac{1}{2} C_2 sint - \frac{1}{2} C_3 e^{\sqrt{3}t} - \frac{1}{2} C_4 e^{-\sqrt{3}t}$$

III –Chapter. Laplace transform

1-§. Laplace transform

Laplace transform – an integral transformation connecting the function $F(s)$ of a complex variable (image) with the function $f(x)$ of a real variable. With this help, the properties of dynamic systems are studied and differential and integral equations are solved. [6].

One of the features of the Laplace transform, which predetermined its wide distribution in scientific and engineering calculations, is that many relations and operations on the originals correspond to simpler relations on their images. Thus, the convolution of two functions is reduced in the image space of the cooperation of multiplication, and linear differential equations become algebraic.

Another Laplace transform is an integral transform, which is closely related to the Fourier transform and has similar properties. It is very often used in engineering disciplines, especially electrical engineering and cybernetics. A complex - valued function $f(x)$ of a real variable t is called original if it is defined at $t \geq 0$, integrable $(0;\ +\infty)$and has exponential order:

$$|f(t)| \leq Ke^{st}, \quad s = const \qquad (1)$$

function

$$F(p) = \int_{0}^{+\infty} e^{-pt} f(t)dt \qquad (2)$$

where p - is a complex parameter, they call it an image (sometimes a transformant) of the original f(t) and write $F(p) = L[f(t)]$. Integral (2) converges absolutely at $Rep > s$, where s – is the constant from (1). Therefore, the image $F(p)$ exists in the half - plane $Rep > s$. The image $F(p)$ in this half – plane is an analytic function of p, which tends to zero at $Rep \to +\infty$ and remains bounded in any half- plane $Rep \geq s_0,\ s_0 > s$.

The following nine theorems provide the basis for the wide applicability of the Laplace transform. The names of the theorems correspond to the operations that are performed on the original functions.

1. **Theorem of addition (***linearity of transformation***)**:

$$L[a_1 f_1(t) + a_2 f_2(t)] = a_1 L[f_1(t)] + a_2 L[f_2(t)]$$

2. **Convolution theorem:**

$$L\left[\int_0^1 f_1(t-\tau)f_2(\tau)d\tau\right] = L[f_1(t)]L[f_2(t)]$$

that is, the convolution in the set of originals corresponds to the usual product of functions in the set of images.

3. **Integration theorem:**

$$L\left[\int_0^1 f(\tau)d\tau\right] = \frac{1}{p}F(p), \qquad F(p) = L[f]$$

Therefore, integration in the area of the originals corresponds to the division of the image into an independent variable.

4. **Differentiation theorem:**

$$L[f^{(n)}(t)] = p^nF(p) - p^{n-1}f_0 - \dots - pf_0^{(n-2)} - f_0^{(n-1)}$$

where $f_0^{(k)} = \lim_{t\to+0} \frac{d^k f(t)}{dt^k}$

5. **Delay theorem:**

$$L[f(t-b)] = e^{-bp}L[f(t)]$$

6. **Similarity theorem:**

The $a > 0$ formula takes place $L[f(at)] = \frac{1}{a}F\left(\frac{p}{a}\right)$

7. **Displacement theorem:**

$$L[e^{-\lambda t}f(t)] = F(p+\boldsymbol{\lambda})$$

8. **Multiplication theorem:**

$$L[t^nf(t)] = (-1)^nF^{(n)}(p)$$

9. **Division theorem:**

If the $\frac{1}{t}f(t)$ Laplace transform is feasible, then the formula holds

$$L\left[\frac{1}{t}f(t)\right] = \int_{p}^{+\infty} F(q)\,dq$$

Example. The formula is fair

$$L(t) = \int_{0}^{+\infty} e^{-pt}dt = \frac{1}{p}.$$

In order to get $L[t^n]$ from here, we take into account that

$$\frac{d^n}{dp^n}\left(\frac{1}{p}\right) = (-1)^n \frac{n!}{p^{n+1}}$$

Then, by multiplication theorem, we obtain that

$$L[t^n] = \frac{n!}{p^{n+1}}.$$

Hence, according to the mixing theorem, we have

$$L[e^{-\lambda t}t^n] = \frac{n!}{(p+\lambda)^{n+1}}$$

Direct Laplace transform

The Laplace transform of a function of a real variable $f(t)$ is a function F(s) of a complex variable $s = \sigma + i\omega,$ such that:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st}f(t)dt$$

The right-hand side of this expression is called the Laplace integral.
The function $f(t)$ is called the original and the image is often denoted as follows:

$$f(t) \risingdotseq F(s) \;\; and \; F(s) \fallingdotseq f(t)$$

Moreover, the image is usually written with a capital letter.

Inverse Laplace transform

By inverse Laplace transform of a function of a complex variable. [6]
If $F(p)$ is an analytical function in the domain

$$Re p \geq s, \quad \lim_{|p|\to\infty} F(p) = 0$$

uniformly relative to argp and

$$\int_{s-i\infty}^{s+i\infty} |F(p)|\,|dp| < +\infty,$$

or $F(p)$ is the image for the function

$$f(t) = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} e^{pt}F(p)dp$$

where $s-$ is some real number.

Of particular importance for applications is the inverse transformation of fractional rational functions with respect to p.

Example. Let's find the function - original

$$F(p) = \frac{1}{p(p+a)}, \quad a \neq 0;$$

$$\frac{1}{p(p+a)} = \frac{1}{ap} - \frac{1}{a(p+a)}$$

The original is found by theorem of addition using the table from

$$f(t) = \frac{1}{a}(1 - e^{-at})$$

2 -§. Application of the Laplace transform to the solution of ordinary differential equations with initial conditions

The great advantage of solving the Cauchy problem for ordinary differential equations using the Laplace transform is that the desired particular solution is obtained directly, rather than fitting the general solution to the given initial conditions. [6].

Let a linear differential equation of the nth order with constant coefficients $(a_0 \neq 0, n > 0)$

$$a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1} y'(t) + a_n y(t) = f(t)$$

and initial conditions

$$y(0) = y_0 ,\ \ y'(0) = y'_0, \ldots, y^{(n-1)}(0) = y_0^{(n-1)}.$$

Application of the Laplace transform to a differential equation, taking into account the differentiation theorem and initial conditions, results in an equation of the form

$$(a_0 p^n + a_1 p^{n-1} + \cdots + a_n) Y(p) = F(p) + y_0(a_0 p^{n-1} + a_1 p^{n-2} + \cdots a_{n-1}) +$$
$$+ y'_0(a_0 p^{n-2} + a_1 p^{n-3} + \cdots + a_{n-2}) + \cdots + y_0^{(n-2)}(a_0 p + a_1) + y_0^{(n-1)} a_0$$

or species

$$Q(p) Y(p) = F(p) + P(p);$$

where in

$Y(p) = L[y(t)]$ - is the image of the desired solution,

$F(p) = L\,[f(t)]$ - is the image of the right side of the original equation, and

$Q(p) = a_0 p^n + a_1 p^{n-1} + \cdots + a_n -$ is the characteristic polynomial of the differential equation. It turns out that

$$Y(p) = F(p)\frac{1}{Q(p)} + \frac{P(p)}{Q(p)}$$

If $y_1(t)$ *and* $y_2(t)$ they are the originals of the functions $\frac{1}{Q(p)}$ *and* $\frac{P(p)}{Q(p)}$ (they can be obtained by decomposing them into elementary fractions), then for the desired solution, according to the convolution theorem, we obtain the formula

$$y(t) = \int_0^t f(t-\tau) y_1(\tau) d\tau + y_2(t)$$

In this case, $F(p)$ does not need to be calculated.
In a completely similar way, you can solve a system of differential equations with constant coefficients. If a system of equations is given

$$y_1'(t) + a_{11}y_1(t) + \cdots + a_{1n}y_n(t) = f_1(t)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_n'(t) + a_{n1}y_1(t) + \cdots + a_{nn}y_n(t) = f_n(t)$$

initial conditions $y_1(0),\ y_2(0),\ \ldots,\ y_n(0)$, then the Laplace transform transforms it into a system of n linear algebraic equations with respect to the n desired images $Y_1(p), Y_2(p), \ldots, Y_n(p)$:

$$(p + a_{11})Y_1(p) + a_{12}Y_2(p) + \cdots + a_{1n}Y_n(p) = F_1(p) + y_1(0),$$

$$a_{21}Y_1(p) + (p + a_{22})Y_2(p) + \cdots + a_{2n}Y_n(p) = F_2(p) + y_2(0),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}Y_1(p) + a_{n2}Y_2(p) + \cdots + (p + a_{nn})Y_n(p) = F_n(p) + y_n(0)$$

The solutions $Y_1(p),\ Y_2(p),\ \ldots,\ Y_n(p)$ to this system must then be inversely transformed in order to obtain solutions $y_1(t),\ y_2(t),\ \ldots,\ y_n(t)$ to the original Cauchy problem.

Example 1.

$$y'(t) + 2y(t) = f(t),$$

where $f(t) = 2[(t + 1)e^{t^2} + (1 + 2t)], \quad y(0) = 1.$

The Laplace transform leads to the equation

$$pY(p) - 1 + 2Y(p) = L[f(t)]$$

where

$$Y(p) = \frac{1}{p + 2} + \frac{L[f]}{p + 2}$$

According to the table of the inverse Laplace transformation and convolution theorem, we obtain that

$$y(t) = e^{-2t} + 2\int_0^t e^{-2(t-\tau)}[(\tau + 1)e^{\tau^2} + (1 + 2\tau)]d\tau$$

The calculation of the integral completes the solution of the problem:

$$y(t) = e^{t^2} + 2t$$

Example 2. $y^{(4)}(t) + 2y'''(t) + 2y''(t) + 2y'(t) + y(t) = 0,$

where $y(0) = y'(0) = 0,\ y''(0) = -2,\quad y'''(0) = 4.$

Trans formation Laplasplants

$$(p^4 + 2p^3 + 2p^2 + 2p + 1)Y(p) = -2p$$

The characteristic polynomial has a root of -1 times two and simple roots $\pm i$. Consequently,

$$Y(p) = \frac{-2p}{(p^2 + 1)(p + 1)^2}$$

The decomposition into elementary fractions has the form

$$\frac{-2p}{(p^2 + 1)(p + 1)^2} = \frac{A}{p + 1} + \frac{B}{(p + 1)^2} + \frac{Cp + D}{p^2 + 1}$$

Comparing the coefficients, we find $A = 0,\ B = 1,\ C = -1,\ D = 0,$ that is

$$Y(p) = \frac{1}{(p + 1)^2} - \frac{1}{p^2 + 1}$$

The table results in

$$y(t) = te^{-t} - sint.$$

Example 3. Solve the Cauchy problem for the system

$$y_1'(t) + y_2(t) = e^t,\quad y_1(0) = 1$$

$$y_2'(t) - y_1(t) = -e^t,\quad y_2(0) = 1$$

Since the right parts of the system have a simple form, it can be easily transformed:

$$pY_1(p) + Y_2(p) = \frac{1}{p - 1} + 1$$

$$pY_2(p) - Y_1(p) = -\frac{1}{p - 1} + 1$$

that is

$$pY_1(p) + Y_2(p) = \frac{p}{p-1}$$

$$pY_2(p) - Y_1(p) = \frac{p-2}{p-1}$$

From here we get

$$Y_1(p) = \frac{p^2 - p + 2}{(p-1)(p^2+1)},$$

$$Y_2(p) = \frac{p}{p^2+1}$$

Expanding into simple fractions:

$$\frac{p^2 - p + 2}{(p-1)(p^2+1)} = \frac{A}{p-1} + \frac{Bp + C}{p^2+1}$$

and equating the coefficients, we get $A = 1,\ B = 0,\ C = -1$, that is

$$Y_1(p) = \frac{1}{p-1} - \frac{1}{p^2+1}$$

From the table we find

$$y_1(t) = e^t - sint$$

The original $y_2(t)$ is directly indicated in the table:

$$y_2(t) = cost$$

3-§. Table of inverse Laplace transformation of fractional rational functions.

The functions in the table are arranged in increasing order of the denominator. The table is complete up to a denominator of degree 3, and also contains several functions whose denominators are polynomials of degree 4. [6].

Inverse Laplace transform tables

$L[f(t)]$	$f(t)$
$\frac{1}{p}$	1
$\frac{1}{p+a}$	e^{-at}
$\frac{1}{p^2}$	t
$\frac{1}{p(p+a)}$	$\frac{1}{a}(1-e^{-at})$
$\frac{1}{(p+a)(p+b)}$	$\frac{1}{b-a}(e^{-at}-e^{-bt})$
$\frac{p}{(p+a)(p+b)}$	$\frac{1}{a-b}(ae^{-at}-be^{-bt})$
$\frac{1}{(p+a)^2}$	te^{-a}
$\frac{p}{(p+a)^2}$	$e^{-at}(1-at)$
$\frac{1}{p^2-a^2}$	$\frac{1}{a}sh(at)$
$\frac{p}{p^2-a^2}$	$ch(at)$
$\frac{1}{p^2+a^2}$	$\frac{1}{a}sin(at)$
$\frac{p}{p^2+a^2}$	$\cos(at)$
$\frac{1}{(p+b)^2+a^2}$	$\frac{1}{a}e^{-at}sin(at)$
$\frac{p}{(p+b)^2+a^2}$	$e^{-bt}(\cos(at)-\frac{b}{a}\sin(at))$
$\frac{1}{p^3}$	$\frac{1}{2}t^2$
$\frac{1}{p^2(p+a)}$	$\frac{1}{a^2}(e^{-at}+at-1)$
$\frac{1}{p(p+a)(p+b)}$	$\frac{1}{ab(a-b)}[(a-b)+be^{-at}-ae^{-bt}]$
$\frac{1}{p(p+a)^2}$	$\frac{1}{a^2}(1-e^{-at}-ate^{-at})$
$\frac{1}{(p+a)(p+b)(p+c)}$	$\frac{1}{(a-b)(b-c)(c-a)}[(c-b)e^{-at}+(a-c)e^{-bt}++(b-a)e^{-ct}]$
$\frac{p}{(p+a)(p+b)(p+c)}$	$\frac{1}{(a-b)(b-c)(c-a)}[a(b-c)e^{-at}+b(c-a)e^{-bt}+c(a-b)e^{-ct}]$
$\frac{p^2}{(p+a)(p+b)(p+c)}$	$\frac{1}{(a-b)(b-c)(c-a)}[a^2(c-b)e^{-at}+b^2(a-c)e^{-bt}+c^2(b-a)e^{-ct}]$
$\frac{1}{(p+a)(p+b)^2}$	$\frac{1}{(b-a)^2}(e^{-at}-e^{-bt}-(b-a)te^{-bt})$
$\frac{p}{(p+a)(p+b)^2}$	$\frac{1}{(b-a)^2}\{-ae^{-at}+[a+bt(b-a)e^{-bt}]\}$

$\frac{p^2}{(p+a)(p+b)^2}$	$\frac{1}{(b-a)^2}\{a^2e^{-at}+b(b-2a-b^2t+abt)e^{-bt}\}$
$\frac{1}{(p+a)^3}$	$\frac{t^2}{2}e^{-at}$
$\frac{p}{(p+a)^3}$	$e^{-at}t\left(1-\frac{a}{2}t\right)$
$\frac{p^2}{(p+a)^3}$	$e^{-at}\left(1-2at+\frac{a^2}{2}t^2\right)$
$\frac{1}{p[(p+b)^2+a^2]}$	$\frac{1}{a^2+b^2}[1-e^{-bt}(\cos(at)+\frac{b}{a}\sin(at)]$
$\frac{1}{p(p^2+a^2)}$	$\frac{1}{a^2}(1-\cos(at))$
$\frac{1}{(p+a)(p^2+b^2)}$	$\frac{1}{a^2+b^2}[e^{-at}+\frac{a}{b}\sin(bt)-\cos(bt)]$
$\frac{p}{(p+a)(p^2+b^2)}$	$\frac{1}{a^2+b^2}[-ae^{-at}+a\cos(bt)+b\sin(bt)]$
$\frac{p^2}{(p+a)(p^2+b^2)}$	$\frac{1}{a^2+b^2}[a^2e^{-at}-ab\sin(bt)+b^2\cos(bt)]$
$\frac{1}{(p+a)[(p+b)^2+c^2]}$	$\frac{1}{(b-a)^2+c^2}[e^{-at}-e^{-bt}\cos(ct)+\frac{a-b}{c}e^{-bt}\sin(ct)]$
$\frac{p}{(p+a)[(p+b)^2+c^2]}$	$\frac{1}{(b-a)^2+c^2}[-ae^{-at}\mp a\cos(ct)+\frac{ab-b^2-c^2}{c}e^{-bt}\sin(ct)]$
$\frac{1}{(p+a)[(p+b)^2+c^2]}$	$\frac{1}{(b-a)^2+c^2}[a^2e^{-at}+((a-b)^2+c^2-c^2)e^{-bt}\cos(ct)-(ac+b(c-\frac{(a-b)b}{c}))e^{-bt}\sin(ct)]$
$\frac{1}{p^4}$	$\frac{1}{6}t^3$
$\frac{1}{p^3(p+a)}$	$\frac{1}{a^3}-\frac{1}{a^2}t+\frac{1}{2a}t^2-\frac{1}{a^3}e^{-at}$
$\frac{1}{p^2(p+a)(p+b)}$	$-\frac{a+b}{a^2b^2}+\frac{1}{ab}t+\frac{1}{a^2(b-a)}e^{-at}+\frac{1}{b^2(a-b)}e^{-bt}$
$\frac{1}{p^2(p+a)^2}$	$\frac{1}{a^2}t(1+e^{-at})+\frac{2}{a^3}(e^{-at}-1)$
$\frac{1}{(p+a)^2(p+b)^2}$	$\frac{1}{(a-b)^2}[e^{-at}(t+\frac{2}{(a-b)})+e^{-bt}(t-\frac{2}{a-b})]$
$\frac{1}{(p+a)^4}$	$\frac{1}{6}t^3e^{-at}$
$\frac{p}{(p+a)^4}$	$\frac{1}{6}t^2e^{-at}-\frac{a}{6}t^3e^{-at}$
$\frac{1}{(p^2+a^2)(p^2+b^2)}$	$\frac{1}{b^2-a^2}[\frac{1}{a}\sin(at)-\frac{1}{b}\sin(bt)]$
$\frac{p}{(p^2+a^2)(p^2+b^2)}$	$\frac{1}{b^2-a^2}[\cos(at)-\cos(bt)]$
$\frac{p^2}{(p^2+a^2)(p^2+b^2)}$	$\frac{1}{b^2-a^2}[-a\sin(at)+b\sin(bt)]$
$\frac{p^3}{(p^2+a^2)(p^2+b^2)}$	$\frac{1}{b^2-a^2}[-a^2\cos(at)+b^2\cos(bt)]$
$\frac{1}{(p^2+a^2)^2}$	$\frac{1}{2a^2}[\frac{1}{a}\sin(at)-t\cos(at)]$
$\frac{p}{(p^2+a^2)^2}$	$\frac{1}{2a}t\sin(at)$

$\frac{p^2}{(p^2+a^2)^2}$	$\frac{1}{2a}(\sin(at) + at\cos(at))$
$\frac{p^3}{(p^2+a^2)^2}$	$\frac{1}{2}[2\cos(at) - at\sin(at)]$
$\frac{1}{[(p+b)^2+a^2]^2}$	$\frac{e^{-bt}}{2a^2}[\frac{1}{a}\sin(at) - t\cos(at)]$
$\frac{1}{p^2(p^2+a^2)}$	$\frac{1}{a^2}(t - \frac{1}{a}\sin(at))$

4-§. Operator method for solving ordinary differential equations

This method consists of passing from a differential equation to an auxiliary algebraic equation through an integral transformation.

The Laplace transform is often used as an integral transform.

$$F(p) = L\{f(t)\} = \int_0^{+\infty} e^{-pt} f(t)dt$$

Information on the conditions under which the image $F(p)$ exists, on the properties of the Laplace transform, and the table of images can be found in the section on integral transformations.

Application of the operator method to solving linear differential equations with constant coefficients.

Let it be necessary to find a solution to the Cauchy problem

$$Q_n\left(\frac{d}{dt}\right) y = f(t)$$

where

$$Q_n\left(\frac{d}{dt}\right) y = \left(a_0\frac{d^n}{dt^n} + a_1\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1}\frac{d}{dt} + a_0\right) y(t)$$

$y(0) = y_0, y'(0) = y'_0, \ldots, y^{(n-1)}(0) = y_0^{(n-1)}, a_i \ (i = 0,1,..,n)$– permanent.

Using the Laplace transform, we reduce this equation, using the notation $L\{y(t)\} = Y(p)$ *and* $L\{f(t)\} = F(p)$, to the auxiliary equation

$$Q_n(p)Y(p) = a_0\left(p^{n-1}y_0 + p^{n-2}y'_0 + \cdots + py_0^{(n-2)} + y_0^{(n-1)}\right) +$$

$$+a_1\left(p^{n-2}y_0 + p^{n-3}y'_0 + \cdots + py_0^{(n-3)} + y_0^{(n-2)}\right) + \cdots + a_{n-1}y_0 + F(p)$$

or

$$Q_n(p)Y(p) = M(p) + F(p)$$

where $Q_n(p) = a_0p^n + a_1p^{n-1} + \cdots + a_n$. The solution to the resulting equation has the form

$$Y(p) = \frac{M(p)}{Q_n(p)} + \frac{F(p)}{Q_n(p)}$$

The inverse transformation carried out only for the second term gives a solution to the differential equation with zero initial values.

Example. $y''' - 3y'' + y' - 3y = 6e^{3t}$; $y(0) = 1, y'(0) = 0, y''(0) = 1$.
In this case

$$F(p) = L(6e^{3t}) = \frac{6}{p-3}$$

and the solution to the auxiliary equation has the form

$$Y(p) = \frac{p^2 - 3p + 2}{p^3 - 3p^2 + p - 3} + \frac{6}{(p-3)(p^3 - 3p^2 + p - 3)}$$

To calculate the inverse transformation $Y(p)$, we expand the right-hand side into simple fractions:

$$Y(p) = \frac{1}{25}\left(\frac{29p - 3}{p^2 + 1} + \frac{15}{(p-3)^2} - \frac{4}{p-3}\right)$$

Using the Laplace transformation table, we find the original in this image:

$$y(t) = \frac{3}{5}te^{3t} - \frac{4}{25}e^{3t} + \frac{29}{25}\cos t - \frac{3}{25}\sin t. \quad ■$$

Dan problem Cauchy

$$\sum_{k=1}^{n} a_{ik}y'_k(t) + \sum_{k=1}^{n} b_{ik}y(t) = f_i(t) \qquad (i = 1,2,\ldots,n)$$

$$y_i(0) = y_{i0} \ (i = 1,2,\ldots,n)$$

under the condition $\det(a_{ik}) \neq 0$. (Applying the Laplace transform to this system, we obtain for the transformed functions $Y_i(p) = L(y_i(t))$ the system

$$\sum_{k=1}^{n}(pa_{ik} + b_{ik})Y_i(p) = F_i(p) + \sum_{k=1}^{n} a_{ik}y_{k0}$$

where $F_i(p) = L(f_i(t))$. From this auxiliary system we find $Y_k(p)$, the inverse transformation and thus obtain the solution to the Cauchy problem.
Example. Find the general solution of the system

$$y'_1 - y_1 + y_2 = t, \qquad y'_2 - 4y_1 + 3y_2 = 2$$

Let us assume $y_1(0) = C_1$, $y_2(0) = C_2$, and solve the auxiliary system

$$(p-1)Y_1(p) + Y_2(p) = \frac{1}{p^2} + C_1$$

$$-4Y_1(p) + (p+3)Y_2(p) = \frac{2}{p} + C_2$$

Relatively $Y_1(p)$ *and* $Y_2(p)$; we get that

$$Y_1(p) = \frac{C_1p^3 + (3C_1 - C_2)p^2 - p + 3}{p^3(p+1)^2}$$

$$Y_2(p) = \frac{C_2p^3 + (2 - C_2 + 4C_1)p^2 - 2p + 4}{p^3(p+1)^2}$$

Carrying out the inverse transformation, we obtain the general solution

$$y_1(t) = 3t - 7 + e^{-t}((4 + 2C_1 - C_2)t + 7 + C_1)$$

$$y_2(t) = 4t - 10 + e^{-t}((8 - 2C_2 + 4C_1)t + 10 + C_2)$$

5 - §. Application of operational calculus to the solution of some differential equations

If given a linear differential equation of nth order with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = f(t)$$

the right side of which $f(t)$ is the original, then the solution of this equation, satisfying arbitrary initial conditions of the form $y(0) = y_0$, $y'(0) = y'_0$, ... , $y^{(n-1)}(0) = y_0^{(n-1)}$ (that is, the solution of the Cauchy problem posed for this equation, with initial conditions at $t = 0$), serves as the original. Denoting the image of this solution by $\bar{y}(p)$, we find the image of the left side of the original differential equation and, equating it to the image of the function $f(t)$, we arrive at the so-called representing equation, which is always a linear algebraic equation with respect to $\bar{y}(p)$. Having determined the formula of this equation $\bar{y}(p)$, we find the original $y(t)$.

Example 1. Solve the differential equation $y'' - 2y' - 3y = e^{3t}$, if $y(0) = 0$, $y'(0) = 0$.

Solution. Let's move on to the images:

$$p^2\bar{y} - p \cdot y(0) - y'(0) - 2\,(p\bar{y} - y(0)) - 3\bar{y} = \frac{1}{p-3}$$

or

$$p^2\bar{y} - 2p\bar{y} - 3\bar{y} = \frac{1}{p-3};\ \bar{y} = \frac{1}{(p+1)(p-3)^2}$$

Let's expand the rational fraction into simpler fractions:

$$\frac{1}{(p+1)(p-3)^2} = \frac{A}{(p-3)^2} + \frac{B}{p-3} + \frac{C}{p+1}$$

$$1 = A(p+1) + B(p-3)(p+1) + C(p-3)^2$$

Putagayar $p = -1$, we get $1 = 16C$, that is $C = 1/16$; $p = 3$ we have $1 = 4A$, that is $A = \frac{1}{4}$. Comparing the coefficients p^2, we get $0 = B + C$, that is $B = -C = \frac{1}{16}$. There fore,

$$\bar{y} = \frac{1}{4(p-3)^2} - \frac{1}{16(p-3)} + \frac{1}{16(p+1)}$$

where

$$y = \frac{1}{4}te^{3t} - \frac{1}{16}e^{3t} + \frac{1}{16}e^{-t}\ . \quad ■$$

Example 2. Solve a system of equations

$$\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = 2x + y + 1 \end{cases}$$

if $x(0) = 0,\ y(0) = 5)$.

Solution. Moving on to the images, we have

$$\begin{cases} p \cdot \bar{x}(p) = \bar{x}(p) + 2\bar{y}(p) \\ p \cdot \bar{y}(p) - 5 = 2\bar{x}(p) + \bar{y}(p) + \dfrac{1}{p} \end{cases}$$

Solving this system for relative $\bar{x}$ *and* $\bar{y}$, we get

$$\bar{x}(p) = \frac{10p + 2}{p(p+1)(p-3)}, \quad \bar{y}(p) = \frac{5p^2 - 4p - 1}{p(p+1)(p-3)}$$

$$u(p) = 10p + 2, \qquad v(p) = p^3 - 2p^2 - 3p, \qquad v'(p) = 3p^2 - 4p - 3$$

$$p_1 = 0,\ p_2 = -1,\ p_3 = 3$$

$$\frac{u(p_1)}{v'(p_1)} = \frac{u(0)}{v'(0)} = -\frac{2}{3}, \qquad \frac{u(p_2)}{v'(p_2)} = \frac{u(-1)}{v'(-1)} = -2, \qquad \frac{u(p_3)}{v'(p_3)} = \frac{u(3)}{v'(3)} = \frac{8}{3}$$

Thus,

$$x = -\frac{2}{3} - 2e^{-t} + \frac{8}{3}e^{3t}.$$

Similarly we find

$$y = \frac{1}{3} + 2e^{-t} + \frac{8}{3}e^{3t}$$

IV- chapter.
Examples for independent work

1. Integrate differential equations with separable variables. [3]

1) $ydx - xdy = 0$ answer. $Y = Cx$

2) $(1 + u)vdu + (1 - v)udv = 0$ answer. $ln\, uv + u - v = C$

3) $(1 + y)\, dx - (1 - x)\, dy = 0$ answer. $(1 + y)(1 - x) = C$

4) $(t^2 - xt^2)\frac{dx}{dt} + x^2 + tx^2 = 0$ answer. $\frac{t+x}{tx} + ln\frac{x}{t} = C$

5) $(y - a)\, dx + x^2dy = 0$ answer. $y - a = Ce^{\frac{1}{x}}$

6) $z\, dt - (t^2 - a^2)dz = 0$ answer. $z^{2a} = C\frac{t-a}{t+a}$

7) $\frac{dx}{dy} = \frac{1+x^2}{1+y^2}.$ answer. $x = \frac{y+C}{1-Cy}$

8) $(1 + s^2)\, dt - \sqrt{t}ds = 0$ *answer.* $2\sqrt{t} - arctgs = C$

9) $dp + ptg\theta d\theta = 0$ answer. $p = C\, cos\theta$

10) $(1 + x^2)dy - \sqrt{1 - y^2}dx = 0$ answer. $arcsiny - arctgx = C$

2. Integrate the following homogeneous differential equations:

11) $(y - x)\, dx + (y + x)\, dy = 0$ answer. $y^2 + 2xy - x^2 = C$

12) $(x + y)\, dx + xdy = 0$ answer. $x^2 + 2xy = C$

13) $(x + y)dx + (y - x)\, dy = 0$ answer. $ln\sqrt{x^2 + y^2} - arctg\frac{y}{x} = C$

14) $xdy - ydx = \sqrt{x^2 + y^2}\, dx$ answer. $1 + 2Cy - C^2x^2 = 0$

15) $(8y + 10x)\, dx + (5y + 7x)\, dy = 0$ answer. $(x + y)^2(2x + y)^3 = C$

16) $(t - s)\, dt + t\, ds = 0$ answer. $te^{\frac{s}{t}} = C$ and $s = tln\frac{C}{t}$

17) $xy^2dy = (x^3 + y^3)\, dx$ answer. $y = x\sqrt[3]{3lnCx}$

18) $(2\sqrt{st} - s)\,dt + t\,ds = 0$ *answer.* $e^{\sqrt{\frac{s}{t}}} = C$ and $s = t\ln^2\frac{C}{t}$

3. Integrate differential equations reduced to homogeneous ones:

19) $(3y - 7x + 7)dx - (3x - 7y - 3)dy = 0$
answer. $(x + y - 1)^5(x - y - 1)^2 = C$

20) $(x + 2y + 1)\,dx - (2x + 4y + 3)\,dy = 0$
answer. $\ln(4x + 8y + 5) + 8y - 4x = C$

21) $(x + 2y + 1)\,dx - (2x - 3)\,dy = 0$ answer. $\ln(2x - 3) - \frac{4y+5}{2x-3} = C$

4. Integrate the following linear differential equations:

22) $y' - \frac{2y}{x+1} = (x + 1)^3$ *answer.* $2y = (x + 1)^4 + C(x + 1)^2$

23) $y' - a\frac{y}{x} = \frac{x+1}{x}$ *answer.* $y = Cx^a + \frac{x}{1-a} - \frac{1}{a}$

24) $(x - x^3)\,y' + (2x^2 - 1)y - ax^3 = 0$ answer. $y = ax + Cx\sqrt{1 - x^2}$

25) $\frac{ds}{dt}\cos t + s\sin t = 1$ answer. $s = \sin t + C\cos t$

26) $\frac{ds}{dt} + s\cos t = \frac{1}{2}\sin 2t$ *answer.* $s = \sin t - 1 + Ce^{-\sin t}$

27) $y' + \frac{n}{x}y = \frac{a}{x^n}$ *answer.* $x^n y = ax + C$

28) $y' + y = e^{-x}$ answer. $e^x y = x + C$

29) $y' + \frac{1-2x}{x^2}y - 1 = 0$ *answer.* $y = x^2(1 + Ce^{\frac{1}{x}})$

30) $y' - \frac{n}{x}y = e^x x^n$ *answer.* $y = x^n(e^x + C)$

5. Integrate the Bernoulli equations:

31) $y' + xy = x^3y^3$ answer. $y^2(x^2 + 1 + Ce^{x^2}) = 1$

32) $(1 - y^2)\, y' - xy - axy^2 = 0$ answer. $(C\sqrt{1-x^2} - a)y = 1$

33) $3y^2y' - ay^3 - x - 1 = 0$ answer. $a^2y^3 = Ce^{ax} - a(x+1) - 1$

34) $y'\,(x^3y^3 + xy) = 1$ answer. $x[(2-y^2)e^{\frac{y^2}{2}}] + C = e^{\frac{y^2}{2}}$

35) $(ylnx - 2)\, ydx = xdy$ answer. $y(Cx^2 + lnx^2 + 1) = 4$

36) $y - y'\, cosx = y^2cosx(1 - sinx)$ answer. $y = \frac{tgx+secx}{sinx+C}$

6. Integrate the following equations in total differentials:

37) $(x^2 + y)dx + (x - 2y)dy = 0$ answer. $\frac{x^3}{3} + yx - y^2 = C$

38) $(y - 3x^2)dx - (4y - x)dy = 0$ answer. $2y^2 - xy + x^3 = C$

39) $(y^3 - x)\, y' = y$ answer. $y^4 = 4xy + C$

40) $\frac{xdx+(2x+y)dy}{(x+y)^2} = 0$ answer. $\ln(x+y) - \frac{x}{x+y} = C$

41) $(\frac{1}{x^2} + \frac{3y^2}{x^4})\, dx = \frac{2ydy}{x^3}$ answer. $x^2 + y^2 = Cx^3$

42) $\frac{x^2dy - y^2dx}{(x-y)^2} = 0$ answer. $\frac{xy}{x-y} = C$

43) $xdx + ydy = \frac{ydx-xdy}{x^2+y^2}$ answer. $x^2 + y^2 - 2arctg\frac{x}{y} = C$

44) $[\frac{y^2}{(x-y)^2} - \frac{1}{x}]\, dx + [\frac{1}{y} - \frac{x^2}{(x-y)^2}]\, dy = 0$ answer. $ln\frac{y}{x} - \frac{xy}{x-y} = C$

7. Integrate the following equations (Lagrange equations):

45) $y = 2xy' + y'^2$ answer. $x = \frac{C}{3p^2} - \frac{2}{3}p, \quad y = \frac{2C-p^3}{3p}$

46) $y = xy'^2 + y'^2$ answer. $y = \left(\sqrt{x+1} + C\right)^2$. Special solution $y = 0$

47) $y = x(1 + y') + (y')^2$ answer. $x = Ce^{-p} - 2p + 2,$
$y = C(p+1)e^{-p} - p^2 + 2$

48) $y = yy'^2 + 2xy'$ answer. $4Cx = 4C^2 - y^2$

8. Solve the Riccati equation: [6]

49) $y' = y^2 - (2x+1)y + (x^2 + x + 1)$
answer. $z'' + (2x+1)z' + (x^2 + x + 1)z = 0$

50) $y' - 3y^2 = x^{-\frac{8}{5}}$, $m = -\frac{8}{5}$ $k = -2,$ $a = -3,$ $b = 1.$
answer. $\bar{y} - 15\bar{y}^2 = 5$

51) $\frac{dy}{dx} = y^2 + \frac{1}{2x^2}$ answer. $y = \frac{1}{x\left[-1 + tg\left(c - \frac{1}{2}\ln x\right)\right]}$

9. Integrate the data from the Clairaut equation:

52) $y = xy' + y' - y'^2$
answer. $y = Cx + C - C^2$. Special solution : $4y = (x+1)^2$

53) $y = xy' + \sqrt{1 - y'^2}$
answer. $y = Cx + \sqrt{1 - C^2}$. Special solution: $y^2 - x^2 = 1$

54) $y = xy' + y'$ answer. $y = Cx + C$

55) $y = xy' + \frac{1}{y'}$ answer. $y = Cx + \frac{1}{C}$. Special solution: $y^2 = 4x$

56) $y = xy' - \frac{1}{y'^2}$ *answer*. $y = Cx - \frac{1}{C^2}$. Special solution: $y^3 = -\frac{27}{4}x^2$

10. Integrate the following linear differential equations with constant coefficients:

57) $y'' = 9y$ answer. $y = C_1e^{3x} + C_2e^{-3x}$

58) $y'' + y = 0$ answer. $y = A\cos x + B\sin x$

59) $y'' - y' = 0$ answer. $y = C_1 + C_2e^x$

60) $y'' + 12y = 7y'$ answer. $y = C_1e^{3x} + C_2e^{4x}$

61) $y'' - 4y' + 4y = 0$ answer. $y = (C_1 + C_2x)e^{2x}$

62) $y'' + 2y' + 10y = 0$ answer. $y = e^{-x}(A\cos 3x + B\sin 3x)$

63) $4y'' - 12y' + 9y = 0$ answer. $y = (C_1 + C_2x)e^{\frac{3x}{2}}$

64) $y'' + y' + y = 0$ answer. $y = e^{-\frac{x}{2}}[A\cos(\frac{\sqrt{3}}{2}x) + B\sin(\frac{\sqrt{3}}{2}x)]$

11. Integrate the following inhomogeneous linear differential equations (find the general solution):

65) $y'' - 7y' + 12y = x$ answer. $y = C_1 e^{3x} + C_2 e^{4x} + \frac{12x+7}{144}$

66) $s'' - a^2s = t + 1$ answer. $s = C_1 e^{at} + C_2 e^{-at} - \frac{t+1}{a^2}$

67) $y'' + y' - 2y = 8\sin 2x$
answer. $y = C_1 e^x + C_2 e^{-2x} - \frac{1}{5}(6\sin 2x + 2\cos 2x)$

68) $y'' - y = 5x + 2$ answer. $y = C_1e^x + C_2e^{-x} - 5x - 2$

69) $s'' - 2as' + a^2s = e^t(a \neq 1)$ *answer.* $s = C_1 e^{at} + C_2 te^{at} + \frac{e^t}{(a-1)^2}$

70) $y'' + 6y' + 5y = e^{2x}$ answer. $y = C_1 e^{-x} + C_2 e^{-5x} + \frac{1}{21}e^{2x}$

12. Integrate the following systems of equations:

71) $\begin{cases} 4\frac{dx}{dt} - \frac{dy}{dt} + 3x = sint \\ \frac{dx}{dt} + y = cost \end{cases}$

$answer.\ \ x = C_1e^{-t} + C_2e^{-3t}, y = C_1e^{-t} + 3C_2e^{-2t}$

72) $\begin{cases} \frac{d^2y}{dt^2} = x \\ \frac{d^2x}{dt^2} = y \end{cases}$ $answer.\ x = C_1e^t + C_2e^{-t} + C_3cost + C_4sint$

$y = C_1e^t + C_2e^{-t} - C_3cost - C_4sint$

73) $\begin{cases} \frac{d^2x}{dt^2} + \frac{dy}{dt} + x = e^t \\ \frac{dx}{dt} + \frac{d^2y}{dt^2} = 1 \end{cases}$ $answer.\ x = C_1 + C_2t + C_3t^2 - \frac{1}{6}t^3 + e^t$

$y = C_4 - (C_1 + 2C_3)t - \frac{1}{2}(C_2 - 1)t^2 - \frac{1}{3}C_3t^3 + \frac{1}{24}t^4 - e^t$

74) $\begin{cases} \frac{dy}{dx} = z - y \\ \frac{dz}{dx} = -y - 3z \end{cases}$

$answer.\ y = (C_1 + C_2x)e^{-2x},\qquad z = (C_2 - C_1 - C_2x)e^{-2x}$

75) $\begin{cases} \frac{dx}{dt} = y + z \\ \frac{dy}{dt} = x + z \\ \frac{dz}{dt} = x + y \end{cases}$ $answer.\ \ x = C_1e^{-t} + C_2e^{2t},\quad y = C_3e^{-t} + C_2e^{2t}$

$z = -(C_1 + C_3)e^{-t} + C_2e^{2t}$

76) $\begin{cases} \frac{dy}{dx} = 1 - \frac{1}{z} \\ \frac{dz}{dx} = \frac{1}{y-x} \end{cases}$ $answer.\ \ z = C_2e^{C_1x},\ y = x + \frac{1}{C_1C_2}e^{-C_1x}$

77) $\begin{cases} \frac{dy}{dx} = \frac{x}{yz} \\ \frac{dz}{dx} = -\frac{x}{y^2} \end{cases}$ $answer.\ \frac{z}{y} = C_1,\ zy^2 - \frac{3}{2}x^2 = C_2$

78) $\begin{cases} \frac{dy}{dx} + z = 0 \\ \frac{dz}{dx} + 4y = 0 \end{cases}$

$answer.\ y = C_1e^{2x} + C_2e^{-2x} z = -2(C_1e^{2x} - C_2e^{-2x})$

13. Application of operational calculus to the solution of some differential equations [3]

Solve differential equations:

79) $y' - 2y = 0;\ y(0) = 1.$ $answer.\quad y = e^{2t}$

80) $y' + y = e^t;\ y(0) = 0.$ $answer.\quad y = sht$

81) $y'' - 9y = 0;\ y(0) = y'(0) = 0.$ $answer.\quad y = 0$

82) $y'' + y' - 2y = e^t;\ y(0) = -1; y'(0) = 0.$
answer. $y = \frac{1}{3}te^t - \frac{7}{9}e^t - \frac{2}{9}e^{-2t}$

83) $y''' - 6y'' + 11y' - 6y = 0;\quad y(0) = 0, y'(0) = 1, y''(0) = 0$

answer. $y = -\frac{5}{2}e^t + 4e^{2t} - \frac{3}{2}e^{3t}$

Solve systems of equations:

84) $\begin{cases} \frac{dx}{dt} = 2y \\ \frac{dy}{dt} = 2x \end{cases}$ $x(0) = 2,\ y(0) = 2.$

answer. $x = \frac{5}{2}e^{2t} - \frac{1}{2}e^{-2t},\quad y = \frac{5}{2}e^{2t} - \frac{1}{2}e^{-2t}$

85) $\begin{cases} \frac{dx}{dt} = 3x + 4y \\ \frac{dy}{dt} = 4x - 3y \end{cases}$ $x(0) = 1,\quad y(0) = 1.$

answer. $x = \frac{6}{5}e^{5t} - \frac{1}{5}e^{-5t},\quad y = \frac{3}{5}e^{5t} + \frac{2}{5}e^{-5t}$

Self - test questions

1. What are the basic concepts of a differential equation?

2. What is a differential equation?
3. Tell me the general concepts about first order differential equations?
4. How can you distinguish between equations with separated and separable variables?
5. What are the types of homogeneous first order equations?
6. Which equations are considered homogeneous?
7. Which equations are considered first order linear equations?
8. Show Bernoulli equation
9. What equations are considered equations in total differentials?
10. What are integrating factors?
11. Show Clairaut's equation?
12. Which differential equations are considered to be of higher order?
13. What are the general properties of a linear homogeneous equation?
14. Show linear homogeneous equations of the second order with constant coefficients.
15. What equations are called linearly independent?
16. Distinguish between an inhomogeneous equation and a homogeneous equation.
17. What are the types of inhomogeneous second-order linear equations with constant coefficients?
18. Inhomogeneous linear equations of higher orders
19. Which systems are called normal?
20. What equations are called characteristic equations?

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Brief information about the author

Kulmirzayeva Gulrabo Abduganiev was born on February 19.02.1972 in the

Jomboy district of the Samarkand region. In 1989 she graduated from the Faculty of Mechanics and Mathematics of Samarkand State University. Since 1994, she began working at the Samarkand College of Finance and Economics. She started teaching computer science, then taught statistics. Since 2000, she began teaching "Mathematics" at the "Exact Sciences" department. Since 2014, she has worked as a senior teacher at the Department of Higher Mathematics at Samarkand Universitete of architecture and construction. During her scientific and methodological activities at SamGASI, she published more than 40 scientific articles and theses, more than 8 guidelines and manuals.

There are 2 patents. She won the "Best Invention and Utility Model" category of the "Woman Inventor – 2022" competition as part of the "100 Women Inventors of the Region" week. In 2022, she took part in the "100 New Faces 2022" competition, held in Kazakhstan, and received a diploma, certificate, first degree badge and breast medals. On January 20, 2023, she was awarded a diploma and badge of the Scientific Research Center for achievements in the field of science and innovation. On April 20, 2023, he was awarded a diploma and badge from the research center "Innovation Promoter" in the field of science and innovation, specializing in "Innovation Promoter." She took part in the international competition of scientific and pedagogical personnel

"Best Researcher - 2023" and was awarded a 2nd degree diploma and a badge.

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